FUNCTIONS DEFINED BY IMPROPER INTEGRALS

William F. Trench
Trinity University

LibreTexts™

Trinity University

Book: Functions Defined by Improper Integrals (Trench)

William F. Trench

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.

LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

The LibreTexts name, logo, and book covers are trademarked by LibreTexts, Inc. (a 501(c)(3) not-for-profit organization) and protected against unauthorized use.

This text was compiled on 08/01/2026

- **InfoPage** by Delmar Larsen is licensed Public Domain.

TABLE OF CONTENTS

Licensing

1: Chapters

- 1.1: Introduction to Improper Functions
- 1.2: Preparation
- 1.3: Uniform convergence of improper integrals
- 1.4: Absolutely Uniformly Convergent Improper Integrals
- 1.5: Dirichlet's Tests
- 1.6: Consequences of Uniform Convergence
- 1.7: Applications to Laplace transforms
- 1.8: Improper Functions (Exercises)

Index

Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Chapters

1.1: Introduction to Improper Functions
1.2: Preparation
1.3: Uniform convergence of improper integrals
1.4: Absolutely Uniformly Convergent Improper Integrals
1.5: Dirichlet's Tests
1.6: Consequences of Uniform Convergence
1.7: Applications to Laplace transforms
1.8: Improper Functions (Exercises)

This page titled 1: Chapters is shared under a not declared license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

1.1: Introduction to Improper Functions

In Section 7.2 (pp. 462–484) we considered functions of the form

$$F(y) = \int_a^b f(x,y)\,dx, \quad c \le y \le d. \tag{1.1.1}$$

We saw that if f is continuous on $[a,b] \times [c,d]$, then F is continuous on $[c,d]$ (Exercise 7.2.3, p. 481) and that we can reverse the order of integration in

$$\int_c^d F(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy \tag{1.1.2}$$

to evaluate it as

$$\int_c^d F(y)\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \tag{1.1.3}$$

(Corollary 7.2.3, p. 466).

Here is another important property of F.

[theorem:1] If f and f_y are continuous on $[a,b] \times [c,d]$, then

$$F(y) = \int_a^b f(x,y)\,dx, \quad c \le y \le d, \tag{1.1.4}$$

is continuously differentiable on $[c,d]$ and $F'(y)$ can be obtained by differentiating [eq:1] under the integral sign with respect to y; that is,

$$F'(y) = \int_a^b f_y(x,y)\,dx, \quad c \le y \le d. \tag{1.1.5}$$

Here $F'(a)$ and $f_y(x,a)$ are derivatives from the right and $F'(b)$ and $f_y(x,b)$ are derivatives from the left.

If y and $y+\Delta y$ are in $[c,d]$ and $\Delta y \ne 0$, then

$$\frac{F(y+\Delta y) - F(y)}{\Delta y} = \int_a^b \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}\,dx. \tag{1.1.6}$$

From the mean value theorem (Theorem 2.3.11, p. 83), if $x \in [a,b]$ and y, $y+\Delta y \in [c,d]$, there is a $y(x)$ between y and $y+\Delta y$ such that

$$f(x,y+\Delta y) - f(x,y) = f_y(x,y)\Delta y = f_y(x,y(x))\Delta y + (f_y(x,y(x)) - f_y(x,y))\Delta y. \tag{1.1.7}$$

From this and [eq:3],

$$\left| \frac{F(y+\Delta y) - F(y)}{\Delta y} - \int_a^b f_y(x,y)\,dx \right| \le \int_a^b |f_y(x,y(x)) - f_y(x,y)|\,dx. \tag{1.1.8}$$

Now suppose $\epsilon > 0$. Since f_y is uniformly continuous on the compact set $[a,b] \times [c,d]$ (Corollary 5.2.14, p. 314) and $y(x)$ is between y and $y+\Delta y$, there is a $\delta > 0$ such that if $|\Delta| < \delta$ then

$$|f_y(x,y) - f_y(x,y(x))| < \epsilon, \quad (x,y) \in [a,b] \times [c,d]. \tag{1.1.9}$$

This and [eq:4] imply that

$$\left| \frac{F(y+\Delta y - F(y))}{\Delta y} - \int_a^b f_y(x,y)\,dx \right| < \epsilon(b-a) \tag{1.1.10}$$

if y and $y+\Delta y$ are in $[c,d]$ and $0 < |\Delta y| < \delta$. This implies [eq:2]. Since the integral in [eq:2] is continuous on $[c,d]$ (Exercise 7.2.3, p. 481, with f replaced by f_y), F' is continuous on $[c,d]$.

[example:1] Since

$$f(x,y) = \cos xy \quad \text{and} \quad f_y(x,y) = -x \sin xy \tag{1.1.11}$$

$$f(x,y)=\cos xy \quad \text{and} \quad f_y(x,y)=-x\sin xy \tag{1.1.11}$$

are continuous for all (x,y), Theorem [theorem:1] implies that if

$$F(y)=\int_0^\pi \cos xy\,dx, \quad -\infty<y<\infty, \tag{1.1.12}$$

then

$$F'(y)=-\int_0^\pi x\sin xy\,dx, \quad -\infty<y<\infty. \tag{1.1.13}$$

(In applying Theorem [theorem:1] for a specific value of y, we take $R=[0,\pi]\times[-\rho,\rho]$, where $\rho>|y|$.) This provides a convenient way to evaluate the integral in [eq:6]: integrating the right side of [eq:5] with respect to x yields

$$F(y)=\frac{\sin xy}{y}\bigg|_{x=0}^{\pi}=\frac{\sin \pi y}{y}, \quad y\neq 0. \tag{1.1.14}$$

Differentiating this and using [eq:6] yields

$$\int_0^\pi x\sin xy\,dx=\frac{\sin \pi y}{y^2}-\frac{\pi\cos \pi y}{y}, \quad y\neq 0. \tag{1.1.15}$$

To verify this, use integration by parts.

We will study the continuity, differentiability, and integrability of

$$F(y)=\int_a^b f(x,y)\,dx, \quad y\in S, \tag{1.1.16}$$

where S is an interval or a union of intervals, and F is a convergent improper integral for each $y\in S$. If the domain of f is $[a,b)\times S$ where $-\infty<a<b\le\infty$, we say that F is *pointwise convergent on* S or simply *convergent on* S, and write

$$\int_a^b f(x,y)\,dx=\lim_{r\to b-}\int_a^r f(x,y)\,dx \tag{1.1.17}$$

if, for each $y\in S$ and every $\epsilon>0$, there is an $r=r_0(y)$ (which also depends on ϵ) such that

$$\left|F(y)-\int_a^r f(x,y)\,dx\right|=\left|\int_r^b f(x,y)\,dx\right|<\epsilon, \quad r_0(y)\le y<b. \tag{1.1.18}$$

If the domain of f is $(a,b]\times S$ where $-\infty\le a<b<\infty$, we replace [eq:7] by

$$\int_a^b f(x,y)\,dx=\lim_{r\to a+}\int_r^b f(x,y)\,dx \tag{1.1.19}$$

and [eq:8] by

$$\left|F(y)-\int_r^b f(x,y)\,dx\right|=\left|\int_a^r f(x,y)\,dx\right|<\epsilon, \quad a<r\le r_0(y). \tag{1.1.20}$$

In general, pointwise convergence of F for all $y\in S$ does not imply that F is continuous or integrable on $[c,d]$, and the additional assumptions that f_y is continuous and $\int_a^b f_y(x,y)\,dx$ converges do not imply [eq:2].

[example:2] The function

$$f(x,y)=ye^{-|y|x} \tag{1.1.21}$$

is continuous on $[0,\infty)\times(-\infty,\infty)$ and

$$F(y)=\int_0^\infty f(x,y)\,dx=\int_0^\infty ye^{-|y|x}\,dx \tag{1.1.22}$$

converges for all y, with

$$F(y)=\begin{cases}-1 & y<0,\\ 0 & y=0,\\ 1 & y>0;\end{cases} \tag{1.1.23}$$

therefore, F is discontinuous at $y = 0$.

[example:3] The function

$$f(x, y) = y^3 e^{-y^2 x} \tag{1.1.24}$$

is continuous on $[0, \infty) \times (-\infty, \infty)$. Let

$$F(y) = \int_0^\infty f(x, y)\, dx = \int_0^\infty y^3 e^{-y^2 x}\, dx = y, \quad -\infty < y < \infty. \tag{1.1.25}$$

Then

$$F'(y) = 1, \quad -\infty < y < \infty. \tag{1.1.26}$$

However,

$$\int_0^\infty \frac{\partial}{\partial y}(y^3 e^{-y^2 x})\, dx = \int_0^\infty (3y^2 - 2y^4 x) e^{-y^2 x}\, dx = \begin{cases} 1, & y \neq 0, \\ 0, & y = 0, \end{cases} \tag{1.1.27}$$

so

$$F'(y) \neq \int_0^\infty \frac{\partial f(x, y)}{\partial y}\, dx \quad \text{if} \quad y = 0. \tag{1.1.28}$$

This page titled 1.1: Introduction to Improper Functions is shared under a CC BY-NC-SA 3.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

1.2: Preparation

We begin with two useful convergence criteria for improper integrals that do not involve a parameter. Consistent with the definition on p. 152, we say that f is locally integrable on an interval I if it is integrable on every finite closed subinterval of I.

[theorem:2] Suppose g is locally integrable on $[a,b)$ and denote

$$G(r)=\int_a^r g(x)\,dx,\quad a\le r<b. \tag{1.2.1}$$

Then the improper integral $\int_a^b g(x)\,dx$ converges if and only if, for each $\epsilon>0$, there is an $r_0\in[a,b)$ such that

$$|G(r)-G(r_1)|<\epsilon,\quad r_0\le r,r_1<b. \tag{1.2.2}$$

For necessity, suppose $\int_a^b g(x)\,dx=L$. By definition, this means that for each $\epsilon>0$ there is an $r_0\in[a,b)$ such that

$$|G(r)-L|<\frac{\epsilon}{2}\quad\text{and}\quad |G(r_1)-L|<\frac{\epsilon}{2},\quad r_0\le r,r_1<b. \tag{1.2.3}$$

Therefore

$$\begin{aligned} |G(r)-G(r_1)|&= |(G(r)-L)-(G(r_1)-L)|\\ &\le |G(r)-L|+|G(r_1)-L|<\epsilon,\quad r_0\le r,r_1<b. \end{aligned} \tag{1.2.4}$$

For sufficiency, [eq:9] implies that

$$|G(r)|=|G(r_1)+(G(r)-G(r_1))|<|G(r_1)|+|G(r)-G(r_1)|\le |G(r_1)|+\epsilon, \tag{1.2.5}$$

$r_0\le r\le r_1<b$. Since G is also bounded on the compact set $[a,r_0]$ (Theorem 5.2.11, p. 313), G is bounded on $[a,b)$. Therefore the monotonic functions

$$\overline{G}(r)=\sup\{G(r_1)\,|\,r\le r_1<b\}\quad\text{and}\quad \underline{G}(r)=\inf\{G(r_1)\,|\,r\le r_1<b\} \tag{1.2.6}$$

are well defined on $[a,b)$, and

$$\lim_{r\to b-}\overline{G}(r)=\overline{L}\quad\text{and}\quad \lim_{r\to b-}\underline{G}(r)=\underline{L} \tag{1.2.7}$$

both exist and are finite (Theorem 2.1.11, p. 47). From [eq:9],

$$\begin{aligned} |G(r)-G(r_1)|&= |(G(r)-G(r_0))-(G(r_1)-G(r_0))|\\ &\le |G(r)-G(r_0)|+|G(r_1)-G(r_0)|<2\epsilon, \end{aligned} \tag{1.2.8}$$

so

$$\overline{G}(r)-\underline{G}(r)\le 2\epsilon,\quad r_0\le r,r_1<b. \tag{1.2.9}$$

Since ϵ is an arbitrary positive number, this implies that

$$\lim_{r\to b-}(\overline{G}(r)-\underline{G}(r))=0, \tag{1.2.10}$$

so $\overline{L}=\underline{L}$. Let $L=\overline{L}=\underline{L}$. Since

$$\underline{G}(r)\le G(r)\le\overline{G}(r), \tag{1.2.11}$$

it follows that $\lim_{r\to b-}G(r)=L$.

We leave the proof of the following theorem to you (Exercise [exer:2]).

[theorem:3] Suppose g is locally integrable on $(a,b]$ and denote

$$G(r)=\int_r^b g(x)\,dx,\quad a\le r<b. \tag{1.2.12}$$

Then the improper integral $\int_a^b g(x)\,dx$ converges if and only if, for each $\epsilon>0$, there is an $r_0\in(a,b]$ such that

$$|G(r)-G(r_1)|<\epsilon,\quad a<r,r_1\le r_0. \tag{1.2.13}$$

To see why we associate Theorems [theorem:2] and [theorem:3] with Cauchy, compare them with Theorem 4.3.5 (p. 204)

This page titled 1.2: Preparation is shared under a CC BY-NC-SA 3.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

1.3: Uniform convergence of improper integrals

Henceforth we deal with functions $f = f(x, y)$ with domains $I \times S$, where S is an interval or a union of intervals and I is of one of the following forms:

- $[a, b)$ with $-\infty < a < b \leq \infty$;
- $(a, b]$ with $-\infty \leq a < b < \infty$;
- (a, b) with $-\infty \leq a \leq b \leq \infty$.

In all cases it is to be understood that f is locally integrable with respect to x on I. When we say that the improper integral $\int_a^b f(x, y)\, dx$ has a stated property "on S" we mean that it has the property for every $y \in S$.

[definition:1] If the improper integral

$$\int_a^b f(x, y)\, dx = \lim_{r \to b-} \int_a^r f(x, y)\, dx \tag{1.3.1}$$

converges on S, it is said to converge uniformly (or be uniformly convergent) on S if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$\left| \int_a^b f(x, y)\, dx - \int_a^r f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r < b, \tag{1.3.2}$$

or, equivalently,

$$\left| \int_r^b f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r < b. \tag{1.3.3}$$

The crucial difference between pointwise and uniform convergence is that $r_0(y)$ in [eq:8] may depend upon the particular value of y, while the r_0 in [eq:11] does not: one choice must work for all $y \in S$. Thus, uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

(Cauchy Criterion for Uniform Convergence I) [theorem:4] The improper integral in [eq:10] converges uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$\left| \int_r^{r_1} f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1 < b. \tag{1.3.4}$$

Suppose $\int_a^b f(x, y)\, dx$ converges uniformly on S and $\epsilon > 0$. From Definition [definition:1], there is an $r_0 \in [a, b)$ such that

$$\left| \int_r^b f(x, y)\, dx \right| < \frac{\epsilon}{2} \text{ and } \left| \int_{r_1}^b f(x, y)\, dx \right| < \frac{\epsilon}{2}, \quad y \in S, \quad r_0 \leq r, r_1 < b. \tag{1.3.5}$$

Since

$$\int_r^{r_1} f(x, y)\, dx = \int_r^b f(x, y)\, dx - \int_{r_1}^b f(x, y)\, dx, \tag{1.3.6}$$

[eq:13] and the triangle inequality imply [eq:12].

For the converse, denote

$$F(y) = \int_a^r f(x, y)\, dx. \tag{1.3.7}$$

Since [eq:12] implies that

$$|F(r, y) - F(r_1, y)| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1 < b, \tag{1.3.8}$$

Theorem [theorem:2] with $G(r) = F(r, y)$ (y fixed but arbitrary in S) implies that $\int_a^b f(x, y)\, dx$ converges pointwise for $y \in S$. Therefore, if $\epsilon > 0$ then, for each $y \in S$, there is an $r_0(y) \in [a, b)$ such that

$$\left| \int_r^b f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0(y) \leq r < b. \tag{1.3.9}$$

For each $y \in S$, choose $r_1(y) \geq \max[r_0(y), r_0]$. (Recall [eq:14]). Then

$$\int_r^b f(x,y)\,dx = \int_r^{r_1(y)} f(x,y)\,dx + \int_{r_1(y)}^b f(x,y)\,dx, \tag{1.3.10}$$

so [eq:12], [eq:15], and the triangle inequality imply that

$$\left|\int_r^b f(x,y)\,dx\right| < 2\epsilon, \quad y \in S, \quad r_0 \le r < b. \tag{1.3.11}$$

In practice, we don't explicitly exhibit r_0 for each given ϵ. It suffices to obtain estimates that clearly imply its existence.

[example:4] For the improper integral of Example [example:2],

$$\left|\int_r^\infty f(x,y)\,dx\right| = \int_r^\infty |y|e^{-|y|x} = e^{-r|y|}, \quad y \ne 0. \tag{1.3.12}$$

If $|y| \ge \rho$, then

$$\left|\int_r^\infty f(x,y)\,dx\right| \le e^{-r\rho}, \tag{1.3.13}$$

so $\int_0^\infty f(x,y)\,dx$ converges uniformly on $(-\infty,\rho] \cup [\rho,\infty)$ if $\rho > 0$; however, it does not converge uniformly on any neighborhood of $y = 0$, since, for any $r > 0$, $e^{-r|y|} > \frac{1}{2}$ if $|y|$ is sufficiently small.

[definition:2] If the improper integral

$$\int_a^b f(x,y)\,dx = \lim_{r\to a+} \int_r^b f(x,y)\,dx \tag{1.3.14}$$

converges on S, it is said to converge uniformly (or be uniformly convergent) on S if, for each $\epsilon > 0$, there is an $r_0 \in (a,b]$ such that

$$\left|\int_a^b f(x,y)\,dx - \int_r^b f(x,y)\,dx\right| < \epsilon, \quad y \in S, \quad a < r \le r_0, \tag{1.3.15}$$

or, equivalently,

$$\left|\int_a^r f(x,y)\,dx\right| < \epsilon, \quad y \in S, \quad a < r \le r_0. \tag{1.3.16}$$

We leave proof of the following theorem to you (Exercise [exer:3]).

(Cauchy Criterion for Uniform Convergence II) [theorem:5] The improper integral

$$\int_a^b f(x,y)\,dx = \lim_{r\to a+} \int_r^b f(x,y)\,dx \tag{1.3.17}$$

converges uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0 \in (a,b]$ such that

$$\left|\int_{r_1}^r f(x,y)\,dx\right| < \epsilon, \quad y \in S, \quad a < r, r_1 \le r_0. \tag{1.3.18}$$

We need one more definition, as follows.

[definition:3] Let $f = f(x,y)$ be defined on $(a,b) \times S$, where $-\infty \le a < b \le \infty$. Suppose f is locally integrable on (a,b) for all $y \in S$ and let c be an arbitrary point in (a,b). Then $\int_a^b f(x,y)\,dx$ is said to converge uniformly on S if $\int_a^c f(x,y)\,dx$ and $\int_c^b f(x,y)\,dx$ both converge uniformly on S.

We leave it to you (Exercise [exer:4]) to show that this definition is independent of c; that is, if $\int_a^c f(x,y)\,dx$ and $\int_c^b f(x,y)\,dx$ both converge uniformly on S for some $c \in (a,b)$, then they both converge uniformly on S for every $c \in (a,b)$.

We also leave it you (Exercise [exer:5]) to show that if f is bounded on $[a,b] \times [c,d]$ and $\int_a^b f(x,y)\,dx$ exists as a proper integral for each $y \in [c,d]$, then it converges uniformly on $[c,d]$ according to all three Definitions [definition:1]–[definition:3].

[example:5] Consider the improper integral

$$F(y) = \int_0^\infty x^{-1/2}e^{-xy}\,dx, \tag{1.3.19}$$

which diverges if $y \le 0$ (verify). Definition [definition:3] applies if $y > 0$, so we consider the improper integrals

$$F_1(y) = \int_0^1 x^{-1/2} e^{-xy}\,dx \quad \text{and} \quad F_2(y) = \int_1^\infty x^{-1/2} e^{-xy}\,dx \tag{1.3.20}$$

separately. Moreover, we could just as well define

$$F_1(y) = \int_0^c x^{-1/2} e^{-xy}\,dx \quad \text{and} \quad F_2(y) = \int_c^\infty x^{-1/2} e^{-xy}\,dx, \tag{1.3.21}$$

where c is any positive number.

Definition [definition:2] applies to F_1. If $0 < r_1 < r$ and $y \ge 0$, then

$$\left|\int_r^{r_1} x^{-1/2} e^{-xy}\,dx\right| < \int_{r_1}^r x^{-1/2}\,dx < 2r^{1/2}, \tag{1.3.22}$$

so $F_1(y)$ converges for uniformly on $[0,\infty)$.

Definition [definition:1] applies to F_2. Since

$$\left|\int_r^{r_1} x^{-1/2} e^{-xy}\,dx\right| < r^{-1/2}\int_r^\infty e^{-xy}\,dx = \frac{e^{-ry}}{yr^{1/2}}, \tag{1.3.23}$$

$F_2(y)$ converges uniformly on $[\rho,\infty)$ if $\rho > 0$. It does not converge uniformly on $(0,\rho)$, since the change of variable $u = xy$ yields

$$\int_r^{r_1} x^{-1/2} e^{-xy}\,dx = y^{-1/2}\int_{ry}^{r_1y} u^{-1/2} e^{-u}\,du, \tag{1.3.24}$$

which, for any fixed $r > 0$, can be made arbitrarily large by taking y sufficiently small and $r = 1/y$. Therefore we conclude that $F(y)$ converges uniformly on $[\rho,\infty)$ if $\rho > 0$.

Note that that the constant c in [eq:16] plays no role in this argument.

[example:6] Suppose we take

$$\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2} \tag{1.3.25}$$

as given (Exercise [exer:31]**(b)**). Substituting $u = xy$ with $y > 0$ yields

$$\int_0^\infty \frac{\sin xy}{x}\,dx = \frac{\pi}{2}, \quad y > 0. \tag{1.3.26}$$

What about uniform convergence? Since $(\sin xy)/x$ is continuous at $x = 0$, Definition [definition:1] and Theorem [theorem:4] apply here. If $0 < r < r_1$ and $y > 0$, then

$$\int_r^{r_1} \frac{\sin xy}{x}\,dx = -\frac{1}{y}\left(\frac{\cos xy}{x}\Big|_r^{r_1} + \int_r^{r_1} \frac{\cos xy}{x^2}\,dx\right), \text{ so } \left|\int_r^{r_1} \frac{\sin xy}{x}\,dx\right| < \frac{3}{ry}. \tag{1.3.27}$$

Therefore [eq:18] converges uniformly on $[\rho,\infty)$ if $\rho > 0$. On the other hand, from [eq:17], there is a $\delta > 0$ such that

$$\int_{u_0}^\infty \frac{\sin u}{u}\,du > \frac{\pi}{4}, \quad 0 \le u_0 < \delta. \tag{1.3.28}$$

This and [eq:18] imply that

$$\int_r^\infty \frac{\sin xy}{x}\,dx = \int_{yr}^\infty \frac{\sin u}{u}\,du > \frac{\pi}{4} \tag{1.3.29}$$

for any $r > 0$ if $0 < y < \delta/r$. Hence, [eq:18] does not converge uniformly on any interval $(0,\rho]$ with $\rho > 0$.

This page titled 1.3: Uniform convergence of improper integrals is shared under a CC BY-NC-SA 3.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

1.4: Absolutely Uniformly Convergent Improper Integrals

(Absolute Uniform Convergence I) [definition:4] The improper integral

$$\int_a^b f(x,y)\,dx = \lim_{r\to b-}\int_a^r f(x,y)\,dx \tag{1.4.1}$$

is said to converge absolutely uniformly on S if the improper integral

$$\int_a^b |f(x,y)|\,dx = \lim_{r\to b-}\int_a^r |f(x,y)|\,dx \tag{1.4.2}$$

converges uniformly on S; that is, if, for each $\epsilon>0$, there is an $r_0\in[a,b)$ such that

$$\left|\int_a^b |f(x,y)|\,dx - \int_a^r |f(x,y)|\,dx\right| < \epsilon, \quad y\in S, \quad r_0<r<b. \tag{1.4.3}$$

To see that this definition makes sense, recall that if f is locally integrable on $[a,b)$ for all y in S, then so is $|f|$ (Theorem 3.4.9, p. 161). Theorem [theorem:4] with f replaced by $|f|$ implies that $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S if and only if, for each $\epsilon>0$, there is an $r_0\in[a,b)$ such that

$$\int_r^{r_1} |f(x,y)|\,dx < \epsilon, \quad y\in S, \quad r_0\le r<r_1<b. \tag{1.4.4}$$

Since

$$\left|\int_r^{r_1} f(x,y)\,dx\right| \le \int_r^{r_1} |f(x,y)|\,dx, \tag{1.4.5}$$

Theorem [theorem:4] implies that if $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S then it converges uniformly on S.

[theorem:6] **(** Weierstrass's Test for Absolute Uniform Convergence I) Suppose $M=M(x)$ is nonnegative on $[a,b)$, $\int_a^b M(x)\,dx<\infty$, and

$$|f(x,y)|\le M(x), \quad y\in S, \quad a\le x<b. \tag{1.4.6}$$

Then $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S.

Denote $\int_a^b M(x)\,dx = L<\infty$. By definition, for each $\epsilon>0$ there is an $r_0\in[a,b)$ such that

$$L-\epsilon < \int_a^r M(x)\,dx \le L, \quad r_0<r<b. \tag{1.4.7}$$

Therefore, if $r_0<r\le r_1$, then

$$0\le \int_r^{r_1} M(x)\,dx = \left(\int_a^{r_1} M(x)\,dx - L\right) - \left(\int_a^r M(x)\,dx - L\right) < \epsilon \tag{1.4.8}$$

This and [eq:19] imply that

$$\int_r^{r_1} |f(x,y)|\,dx \le \int_r^{r_1} M(x)\,dx < \epsilon, \quad y\in S, \quad a\le r_0<r<r_1<b. \tag{1.4.9}$$

Now Theorem [theorem:4] implies the stated conclusion.

[example:7] Suppose $g=g(x,y)$ is locally integrable on $[0,\infty)$ for all $y\in S$ and, for some $a_0\ge 0$, there are constants K and p_0 such that

$$|g(x,y)|\le Ke^{p_0x}, \quad y\in S, \quad x\ge a_0. \tag{1.4.10}$$

If $p>p_0$ and $r\ge a_0$, then

$$\begin{aligned}\int_r^\infty e^{-px}|g(x,y)|\,dx = &\int_r^\infty e^{-(p-p_0)x}e^{-p_0x}|g(x,y)|\,dx\\ \le\ & K\int_r^\infty e^{-(p-p_0)x}\,dx = \frac{Ke^{-(p-p_0)r}}{p-p_0},\end{aligned} \tag{1.4.11}$$

so $\int_0^\infty e^{-px}g(x,y)\,dx$ converges absolutely on S. For example, since

$$|x^\alpha \sin xy| < e^{p_0x} \quad \text{and} \quad |x^\alpha \cos xy| < e^{p_0x} \tag{1.4.12}$$

for x sufficiently large if $p_0 > 0$, Theorem [theorem:4] implies that $\int_0^\infty e^{-px}x^\alpha \sin xy\,dx$ and $\int_0^\infty e^{-px}x^\alpha \cos xy\,dx$ converge absolutely uniformly on $(-\infty,\infty)$ if $p > 0$ and $\alpha \geq 0$. As a matter of fact, $\int_0^\infty e^{-px}x^\alpha \sin xy\,dx$ converges absolutely on $(-\infty,\infty)$ if $p > 0$ and $\alpha > -1$. (Why?)

(Absolute Uniform Convergence II) [definition:5] The improper integral

$$\int_a^b f(x,y)\,dx = \lim_{r\to a+}\int_r^b f(x,y)\,dx \tag{1.4.13}$$

is said to converge absolutely uniformly on S if the improper integral

$$\int_a^b |f(x,y)|\,dx = \lim_{r\to a+}\int_r^b |f(x,y)|\,dx \tag{1.4.14}$$

converges uniformly on S; that is, if, for each $\epsilon > 0$, there is an $r_0 \in (a,b]$ such that

$$\left|\int_a^b |f(x,y)|\,dx - \int_r^b |f(x,y)|\,dx\right| < \epsilon, \quad y \in S, \quad a < r < r_0 \leq b. \tag{1.4.15}$$

We leave it to you (Exercise [exer:7]) to prove the following theorem.

[theorem:7] **(**Weierstrass's Test for Absolute Uniform Convergence II) Suppose $M = M(x)$ is nonnegative on $(a,b]$, $\int_a^b M(x)\,dx < \infty$, and

$$|f(x,y)| \leq M(x), \quad y \in S, \quad x \in (a,b]. \tag{1.4.16}$$

Then $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S.

[example:8] If $g = g(x,y)$ is locally integrable on $(0,1]$ for all $y \in S$ and

$$|g(x,y)| \leq Ax^{-\beta}, \quad 0 < x \leq x_0, \tag{1.4.17}$$

for each $y \in S$, then

$$\int_0^1 x^\alpha g(x,y)\,dx \tag{1.4.18}$$

converges absolutely uniformly on S if $\alpha > \beta - 1$. To see this, note that if $0 < r < r_1 \leq x_0$, then

$$\int_{r_1}^r x^\alpha |g(x,y)|\,dx \leq A\int_{r_1}^r x^{\alpha-\beta}\,dx = \frac{Ax^{\alpha-\beta+1}}{\alpha-\beta+1}\Big|_{r_1}^r < \frac{Ar^{\alpha-\beta+1}}{\alpha-\beta+1}. \tag{1.4.19}$$

Applying this with $\beta = 0$ shows that

$$F(y) = \int_0^1 x^\alpha \cos xy\,dx \tag{1.4.20}$$

converges absolutely uniformly on $(-\infty,\infty)$ if $\alpha > -1$ and

$$G(y) = \int_0^1 x^\alpha \sin xy\,dx \tag{1.4.21}$$

converges absolutely uniformly on $(-\infty,\infty)$ if $\alpha > -2$.

By recalling Theorem 4.4.15 (p. 246), you can see why we associate Theorems [theorem:6] and [theorem:7] with Weierstrass.

This page titled 1.4: Absolutely Uniformly Convergent Improper Integrals is shared under a CC BY-NC-SA 3.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

1.5: Dirichlet's Tests

Weierstrass's test is useful and important, but it has a basic shortcoming: it applies only to absolutely uniformly convergent improper integrals. The next theorem applies in some cases where $\int_a^b f(x,y)\,dx$ converges uniformly on S, but $\int_a^b |f(x,y)|\,dx$ does not.

[theorem:8] (**Dirichlet's Test for Uniform Convergence I**) If g, g_x, and h are continuous on $[a,b)\times S$, then

$$\int_a^b g(x,y)h(x,y)\,dx \tag{1.5.1}$$

converges uniformly on S if the following conditions are satisfied:

$$\lim_{x\to b-}\left\{\sup_{y\in S}|g(x,y)|\right\}=0; \tag{1.5.2}$$

There is a constant M such that

$$\sup_{y\in S}\left|\int_a^x h(u,y)\,du\right|<M,\quad a\le x<b; \tag{1.5.3}$$

$\int_a^b |g_x(x,y)|\,dx$ converges uniformly on S.

If

$$H(x,y)=\int_a^x h(u,y)\,du, \tag{1.5.4}$$

then integration by parts yields

$$\begin{aligned}\int_r^{r_1} g(x,y)h(x,y)\,dx &= \int_r^{r_1} g(x,y)H_x(x,y)\,dx\\ &= g(r_1,y)H(r_1,y)-g(r,y)H(r,y)\\ &\quad -\int_r^{r_1} g_x(x,y)H(x,y)\,dx.\end{aligned} \tag{1.5.5}$$

Since assumption **(b)** and [eq:20] imply that $|H(x,y)|\le M$, $(x,y)\in(a,b]\times S$, Eqn. [eq:21] implies that

$$\left|\int_r^{r_1} g(x,y)h(x,y)\,dx\right|<M\left(2\sup_{x\ge r}|g(x,y)|+\int_r^{r_1}|g_x(x,y)|\,dx\right) \tag{1.5.6}$$

on $[r,r_1]\times S$.

Now suppose $\epsilon>0$. From assumption **(a)**, there is an $r_0\in[a,b)$ such that $|g(x,y)|<\epsilon$ on S if $r_0\le x<b$. From assumption **(c)** and Theorem [theorem:6], there is an $s_0\in[a,b)$ such that

$$\int_r^{r_1}|g_x(x,y)|\,dx<\epsilon,\quad y\in S,\quad s_0<r<r_1<b. \tag{1.5.7}$$

Therefore [eq:22] implies that

$$\left|\int_r^{r_1} g(x,y)h(x,y)\right|<3M\epsilon,\quad y\in S,\quad \max(r_0,s_0)<r<r_1<b. \tag{1.5.8}$$

Now Theorem [theorem:4] implies the stated conclusion.

The statement of this theorem is complicated, but applying it isn't; just look for a factorization $f=gh$, where h has a bounded antderivative on $[a,b)$ and g is "small" near b. Then integrate by parts and hope that something nice happens. A similar comment applies to Theorem 9, which follows.

Example 1.5.1

Add text here.

Solution

Add text here.

Let

$$I(y)=\int_0^\infty \frac{\cos xy}{x+y}\,dx,\quad y>0. \tag{1.5.9}$$

The obvious inequality

$$\left|\frac{\cos xy}{x+y}\right|\le\frac{1}{x+y} \tag{1.5.10}$$

is useless here, since

$$\int_0^\infty \frac{dx}{x+y}=\infty. \tag{1.5.11}$$

However, integration by parts yields

$$\begin{aligned}\int_r^{r_1}\frac{\cos xy}{x+y}\,dx &= \frac{\sin xy}{y(x+y)}\bigg|_r^{r_1}+\int_r^{r_1}\frac{\sin xy}{y(x+y)^2}\,dx\\ &= \frac{\sin r_1y}{y(r_1+y)}-\frac{\sin ry}{y(r+y)}+\int_r^{r_1}\frac{\sin xy}{y(x+y)^2}\,dx.\end{aligned} \tag{1.5.12}$$

Therefore, if $0<r<r_1$, then

$$\left|\int_r^{r_1}\frac{\cos xy}{x+y}\,dx\right|<\frac{1}{y}\left(\frac{2}{r+y}+\int_r^\infty\frac{1}{(x+y)^2}\right)\le\frac{3}{y(r+y)^2}\le\frac{3}{\rho(r+\rho)} \tag{1.5.13}$$

if $y\ge\rho>0$. Now Theorem [theorem:4] implies that $I(y)$ converges uniformly on $[\rho,\infty)$ if $\rho>0$.

We leave the proof of the following theorem to you (Exercise [exer:10]).

[theorem:9] (**Dirichlet's Test for Uniform Convergence II**) If g, g_x, and h are continuous on $(a,b]\times S$, then

$$\int_a^b g(x,y)h(x,y)\,dx \tag{1.5.14}$$

converges uniformly on S if the following conditions are satisfied:

$\lim_{x\to a+}\left\{\sup_{y\in S}|g(x,y)|\right\}=0$;

There is a constant M such that

$$\sup_{y\in S}\left|\int_x^b h(u,y)\,du\right|\le M,\quad a<x\le b; \tag{1.5.15}$$

$\int_a^b|g_x(x,y)|\,dx$ converges uniformly on S.

By recalling Theorems 3.4.10 (p. 163), 4.3.20 (p. 217), and 4.4.16 (p. 248), you can see why we associate Theorems [theorem:8] and [theorem:9] with Dirichlet.

This page titled 1.5: Dirichlet's Tests is shared under a CC BY-NC-SA 3.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

1.6: Consequences of Uniform Convergence

[theorem:10] If $f = f(x,y)$ is continuous on either $[a,b) \times [c,d]$ or $(a,b] \times [c,d]$ and

$$F(y) = \int_a^b f(x,y)\,dx \tag{1.6.1}$$

converges uniformly on $[c,d]$, then F is continuous on $[c,d]$. Moreover,

$$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx. \tag{1.6.2}$$

We will assume that f is continuous on $(a,b] \times [c,d]$. You can consider the other case (Exercise [exer:14]).

We will first show that F in [eq:23] is continuous on $[c,d]$. Since F converges uniformly on $[c,d]$, Definition [definition:1] (specifically, [eq:11]) implies that if $\epsilon > 0$, there is an $r \in [a,b)$ such that

$$\left| \int_r^b f(x,y)\,dx \right| < \epsilon, \quad c \le y \le d. \tag{1.6.3}$$

Therefore, if $c \le y, y_0 \le d]$, then

$$\begin{aligned} |F(y) - F(y_0)| = & \left| \int_a^b f(x,y)\,dx - \int_a^b f(x,y_0)\,dx \right| \\ \le & \left| \int_a^r [f(x,y) - f(x,y_0)]\,dx \right| + \left| \int_r^b f(x,y)\,dx \right| \\ & + \left| \int_r^b f(x,y_0)\,dx \right|, \end{aligned} \tag{1.6.4}$$

so

$$|F(y) - F(y_0)| \le \int_a^r |f(x,y) - f(x,y_0)|\,dx + 2\epsilon. \tag{1.6.5}$$

Since f is uniformly continuous on the compact set $[a,r] \times [c,d]$ (Corollary 5.2.14, p. 314), there is a $\delta > 0$ such that

$$|f(x,y) - f(x,y_0)| < \epsilon \tag{1.6.6}$$

if (x,y) and (x,y_0) are in $[a,r] \times [c,d]$ and $|y - y_0| < \delta$. This and [eq:25] imply that

$$|F(y) - F(y_0)| < (r-a)\epsilon + 2\epsilon < (b-a+2)\epsilon \tag{1.6.7}$$

if y and y_0 are in $[c,d]$ and $|y - y_0| < \delta$. Therefore F is continuous on $[c,d]$, so the integral on left side of [eq:24] exists. Denote

$$I = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy. \tag{1.6.8}$$

We will show that the improper integral on the right side of [eq:24] converges to I. To this end, denote

$$I(r) = \int_a^r \left(\int_c^d f(x,y)\,dy \right) dx. \tag{1.6.9}$$

Since we can reverse the order of integration of the continuous function f over the rectangle $[a,r] \times [c,d]$ (Corollary 7.2.2, p. 466),

$$I(r) = \int_c^d \left(\int_a^r f(x,y)\,dx \right) dy. \tag{1.6.10}$$

From this and [eq:26],

$$I - I(r) = \int_c^d \left(\int_r^b f(x,y)\,dx \right) dy. \tag{1.6.11}$$

Now suppose $\epsilon > 0$. Since $\int_a^b f(x,y)\,dx$ converges uniformly on $[c,d]$, there is an $r_0 \in (a,b]$ such that

$$\left|\int_r^v f(x,y)\,dx\right| < \epsilon, \quad r_0 < r < b, \tag{1.6.12}$$

so $|I - I(r)| < (d-c)\epsilon$ if $r_0 < r < b$. Hence,

$$\lim_{r\to b-}\int_a^r \left(\int_c^d f(x,y)\,dy\right)\,dx = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy, \tag{1.6.13}$$

which completes the proof of [eq:24].

[example:10] It is straightforward to verify that

$$\int_0^\infty e^{-xy}\,dx = \frac{1}{y}, \quad y > 0, \tag{1.6.14}$$

and the convergence is uniform on $[\rho,\infty)$ if $\rho > 0$. Therefore Theorem [theorem:10] implies that if $0 < y_1 < y_2$, then

$$\begin{aligned}\int_{y_1}^{y_2}\frac{dy}{y} = &\int_{y_1}^{y_2}\left(\int_0^\infty e^{-xy}\,dx\right)\,dy = \int_0^\infty\left(\int_{y_1}^{y_2} e^{-xy}\,dy\right)\,dy\\ = &\int_0^\infty \frac{e^{-xy_1}-e^{-xy_2}}{x}\,dx.\end{aligned} \tag{1.6.15}$$

Since

$$\int_{y_1}^{y_2}\frac{dy}{y} = \log\frac{y_2}{y_1}, \quad y_2 \ge y_1 > 0, \tag{1.6.16}$$

it follows that

$$\int_0^\infty \frac{e^{-xy_1}-e^{-xy_2}}{x}\,dx = \log\frac{y_2}{y_1}, \quad y_2 \ge y_1 > 0. \tag{1.6.17}$$

[example:11] From Example [example:6],

$$\int_0^\infty \frac{\sin xy}{x}\,dx = \frac{\pi}{2}, \quad y > 0, \tag{1.6.18}$$

and the convergence is uniform on $[\rho,\infty)$ if $\rho > 0$. Therefore, Theorem [theorem:10] implies that if $0 < y_1 < y_2$, then

$$\begin{aligned}\frac{\pi}{2}(y_2-y_1) = &\int_{y_1}^{y_2}\left(\int_0^\infty \frac{\sin xy}{x}\,dx\right)\,dy = \int_0^\infty\left(\int_{y_1}^{y_2}\frac{\sin xy}{x}\,dy\right)\,dx\\ = &\int_0^\infty \frac{\cos xy_1 - \cos xy_2}{x^2}\,dx.\end{aligned} \tag{1.6.19}$$

The last integral converges uniformly on $(-\infty,\infty)$ (Exercise 10**(h)**), and is therefore continuous with respect to y_1 on $(-\infty,\infty)$, by Theorem [theorem:10]; in particular, we can let $y_1 \to 0+$ in [eq:27] and replace y_2 by y to obtain

$$\int_0^\infty \frac{1-\cos xy}{x^2}\,dx = \frac{\pi y}{2}, \quad y \ge 0. \tag{1.6.20}$$

The next theorem is analogous to Theorem 4.4.20 (p. 252).

[theorem:11] Let f and f_y be continuous on either $[a,b)\times[c,d]$ or $(a,b]\times[c,d]$. Suppose that the improper integral

$$F(y) = \int_a^b f(x,y)\,dx \tag{1.6.21}$$

converges for some $y_0 \in [c,d]$ and

$$G(y) = \int_a^b f_y(x,y)\,dx \tag{1.6.22}$$

converges uniformly on $[c,d]$. Then F converges uniformly on $[c,d]$ and is given explicitly by

$$F(y) = F(y_0) + \int_{y_0}^y G(t)\,dt, \quad c \le y \le d. \tag{1.6.23}$$

Moreover, F is continuously differentiable on $[c,d]$; specifically,

$$F'(y)=G(y),\quad c\le y\le d, \tag{1.6.24}$$

where $F'(c)$ and $f_y(x,c)$ are derivatives from the right, and $F'(d)$ and $f_y(x,d)$ are derivatives from the left.

We will assume that f and f_y are continuous on $[a,b)\times[c,d]$. You can consider the other case (Exercise [exer:15]).

Let

$$F_r(y)=\int_a^r f(x,y)\,dx,\quad a\le r<b,\quad c\le y\le d. \tag{1.6.25}$$

Since f and f_y are continuous on $[a,r]\times[c,d]$, Theorem [theorem:1] implies that

$$F_r'(y)=\int_a^r f_y(x,y)\,dx,\quad c\le y\le d. \tag{1.6.26}$$

Then

$$\begin{aligned} F_r(y) &= F_r(y_0)+\int_{y_0}^y\left(\int_a^r f_y(x,t)\,dx\right)\,dt\\ &= F(y_0)+\int_{y_0}^y G(t)\,dt\\ &\quad +(F_r(y_0)-F(y_0))-\int_{y_0}^y\left(\int_r^b f_y(x,t)\,dx\right)\,dt,\quad c\le y\le d. \end{aligned} \tag{1.6.27}$$

Therefore,

$$\begin{aligned} \left|F_r(y)-F(y_0)-\int_{y_0}^y G(t)\,dt\right| \le\ & |F_r(y_0)-F(y_0)|\\ &+\left|\int_{y_0}^y\int_r^b f_y(x,t)\,dx\right|\,dt. \end{aligned} \tag{1.6.28}$$

Now suppose $\epsilon>0$. Since we have assumed that $\lim_{r\to b-}F_r(y_0)=F(y_0)$ exists, there is an r_0 in (a,b) such that

$$|F_r(y_0)-F(y_0)|<\epsilon,\quad r_0<r<b. \tag{1.6.29}$$

Since we have assumed that $G(y)$ converges for $y\in[c,d]$, there is an $r_1\in[a,b)$ such that

$$\left|\int_r^b f_y(x,t)\,dx\right|<\epsilon,\quad t\in[c,d],\quad r_1\le r<b. \tag{1.6.30}$$

Therefore, [eq:29] yields

$$\left|F_r(y)-F(y_0)-\int_{y_0}^y G(t)\,dt\right|<\epsilon(1+|y-y_0|)\le\epsilon(1+d-c) \tag{1.6.31}$$

if $\max(r_0,r_1)\le r<b$ and $t\in[c,d]$. Therefore $F(y)$ converges uniformly on $[c,d]$ and

$$F(y)=F(y_0)+\int_{y_0}^y G(t)\,dt,\quad c\le y\le d. \tag{1.6.32}$$

Since G is continuous on $[c,d]$ by Theorem [theorem:10], [eq:28] follows from differentiating this (Theorem 3.3.11, p. 141).

[example:12] Let

$$I(y)=\int_0^\infty e^{-yx^2}\,dx,\quad y>0. \tag{1.6.33}$$

Since

$$\int_0^r e^{-yx^2}\,dx=\frac{1}{\sqrt{y}}\int_0^{r\sqrt{y}}e^{-t^2}\,dt, \tag{1.6.34}$$

it follows that

$$I(y)=\frac{1}{\sqrt{y}}\int_0^\infty e^{-t^2}\,dt, \tag{1.6.35}$$

and the convergence is uniform on $[\rho,\infty)$ if $\rho>0$ (Exercise [exer:8]**(i)**). To evaluate the last integral, denote $J(\rho)=\int_0^\rho e^{-t^2}\,dt$; then

$$J^2(\rho)=\left(\int_0^\rho e^{-u^2}\,du\right)\left(\int_0^\rho e^{-v^2}\,dv\right)=\int_0^\rho\int_0^\rho e^{-(u^2+v^2)}\,du\,dv. \tag{1.6.36}$$

Transforming to polar coordinates $r=r\cos\theta$, $v=r\sin\theta$ yields

$$J^2(\rho)=\int_0^{\pi/2}\int_0^\rho re^{-r^2}\,dr\,d\theta=\frac{\pi(1-e^{-\rho^2})}{4},\quad\text{so}\quad J(\rho)=\frac{\sqrt{\pi(1-e^{-\rho^2})}}{2}. \tag{1.6.37}$$

Therefore

$$\int_0^\infty e^{-t^2}\,dt=\lim_{\rho\to\infty}J(\rho)=\frac{\sqrt{\pi}}{2}\quad\text{and}\quad\int_0^\infty e^{-yx^2}\,dx=\frac{1}{2}\sqrt{\frac{\pi}{y}},\quad y>0. \tag{1.6.38}$$

Differentiating this n times with respect to y yields

$$\int_0^\infty x^{2n}e^{-yx^2}\,dx=\frac{1\cdot3\cdots(2n-1)\sqrt{\pi}}{2^ny^{n+1/2}}\quad y>0,\quad n=1,2,3,\dots, \tag{1.6.39}$$

where Theorem [theorem:11] justifies the differentiation for every n, since all these integrals converge uniformly on $[\rho,\infty)$ if $\rho>0$ (Exercise [exer:8](i)).

Some advice for applying this theorem: Be sure to check first that $F(y_0)=\int_a^b f(x,y_0)\,dx$ converges for at least one value of y. If so, differentiate $\int_a^b f(x,y)\,dx$ formally to obtain $\int_a^b f_y(x,y)\,dx$. Then $F'(y)=\int_a^b f_y(x,y)\,dx$ if y is in some interval on which this improper integral converges uniformly.

This page titled 1.6: Consequences of Uniform Convergence is shared under a CC BY-NC-SA 3.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

1.7: Applications to Laplace transforms

The *Laplace transform* of a function f locally integrable on $[0,\infty)$ is

$$F(s)=\int_0^\infty e^{-sx}f(x)\,dx \tag{1.7.1}$$

for all s such that integral converges. Laplace transforms are widely applied in mathematics, particularly in solving differential equations.

We leave it to you to prove the following theorem (Exercise [exer:26]).

[theorem:12] Suppose f is locally integrable on $[0,\infty)$ and $|f(x)|\le Me^{s_0x}$ for sufficiently large x. Then the Laplace transform of F converges uniformly on $[s_1,\infty)$ if $s_1>s_0$.

[theorem:13] If f is continuous on $[0,\infty)$ and $H(x)=\int_0^\infty e^{-s_0u}f(u)\,du$ is bounded on $[0,\infty)$, then the Laplace transform of f converges uniformly on $[s_1,\infty)$ if $s_1>s_0$.

If $0\le r\le r_1$,

$$\int_r^{r_1}e^{-sx}f(x)\,dx=\int_r^{r_1}e^{-(s-s_0)x}e^{-s_0x}f(x)\,dt=\int_r^{r_1}e^{-(s-s_0)t}H'(x)\,dt. \tag{1.7.2}$$

Integration by parts yields

$$\int_r^{r_1}e^{-sx}f(x)\,dt=e^{-(s-s_0)x}H(x)\Big|_r^{r_1}+(s-s_0)\int_r^{r_1}e^{-(s-s_0)x}H(x)\,dx. \tag{1.7.3}$$

Therefore, if $|H(x)|\le M$, then

$$\begin{aligned}\left|\int_r^{r_1}e^{-sx}f(x)\,dx\right|\le\quad & M\left|e^{-(s-s_0)r_1}+e^{-(s-s_0)r}+(s-s_0)\int_r^{r_1}e^{-(s-s_0)x}\,dx\right|\\ \le\quad & 3Me^{-(s-s_0)r}\le 3Me^{-(s_1-s_0)r},\quad s\ge s_1.\end{aligned} \tag{1.7.4}$$

Now Theorem [theorem:4] implies that $F(s)$ converges uniformly on $[s_1,\infty)$.

The following theorem draws a considerably stonger conclusion from the same assumptions.

[theorem:14] If f is continuous on $[0,\infty)$ and

$$H(x)=\int_0^x e^{-s_0u}f(u)\,du \tag{1.7.5}$$

is bounded on $[0,\infty)$, then the Laplace transform of f is infinitely differentiable on (s_0,∞), with

$$F^{(n)}(s)=(-1)^n\int_0^\infty e^{-sx}x^nf(x)\,dx; \tag{1.7.6}$$

that is, the n-th derivative of the Laplace transform of $f(x)$ is the Laplace transform of $(-1)^nx^nf(x)$.

First we will show that the integrals

$$I_n(s)=\int_0^\infty e^{-sx}x^nf(x)\,dx,\quad n=0,1,2,... \tag{1.7.7}$$

all converge uniformly on $[s_1,\infty)$ if $s_1>s_0$. If $0<r<r_1$, then

$$\int_r^{r_1}e^{-sx}x^nf(x)\,dx=\int_r^{r_1}e^{-(s-s_0)x}e^{-s_0x}x^nf(x)\,dx=\int_r^{r_1}e^{-(s-s_0)x}x^nH'(x)\,dx. \tag{1.7.8}$$

Integrating by parts yields

$$\begin{aligned}\int_r^{r_1}e^{-sx}x^nf(x)\,dx=\quad & r_1^ne^{-(s-s_0)r_1}H(r)-r^ne^{-(s-s_0)r}H(r)\\ & -\int_r^{r_1}H(x)\left(e^{-(s-s_0)x}x^n\right)'\,dx,\end{aligned} \tag{1.7.9}$$

where $'$ indicates differentiation with respect to x. Therefore, if $|H(x)| \le M \le \infty$ on $[0,\infty)$, then

$$\left|\int_r^{r_1} e^{-sx}x^n f(x)\,dx\right| \le M\left(e^{-(s-s_0)r}r^n + e^{-(s-s_0)r}r^n + \int_r^\infty |(e^{-(s-s_0)x})x^n)'|\,dx\right). \tag{1.7.10}$$

Therefore, since $e^{-(s-s_0)r}r^n$ decreases monotonically on (n,∞) if $s > s_0$ (check!),

$$\left|\int_r^{r_1} e^{-sx}x^n f(x)\,dx\right| < 3Me^{-(s-s_0)r}r^n, \quad n < r < r_1, \tag{1.7.11}$$

so Theorem [theorem:4] implies that $I_n(s)$ converges uniformly $[s_1,\infty)$ if $s_1 > s_0$. Now Theorem [theorem:11] implies that $F_{n+1} = -F_n'$, and an easy induction proof yields [eq:30] (Exercise [exer:25]).

[example:13] Here we apply Theorem [theorem:12] with $f(x) = \cos ax$ ($a \ne 0$) and $s_0 = 0$. Since

$$\int_0^x \cos au\,du = \frac{\sin ax}{a} \tag{1.7.12}$$

is bounded on $(0,\infty)$, Theorem [theorem:12] implies that

$$F(s) = \int_0^\infty e^{-sx}\cos ax\,dx \tag{1.7.13}$$

converges and

$$F^{(n)}(s) = (-1)^n\int_0^\infty e^{-sx}x^n\cos ax\,dx, \quad s > 0. \tag{1.7.14}$$

(Note that this is also true if $a = 0$.) Elementary integration yields

$$F(s) = \frac{s}{s^2+a^2}. \tag{1.7.15}$$

Hence, from [eq:31],

$$\int_0^\infty e^{-sx}x^n\cos ax = (-1)^n\frac{d^n}{ds^n}\frac{s}{s^2+a^2}, \quad n = 0,1,\dots. \tag{1.7.16}$$

This page titled 1.7: Applications to Laplace transforms is shared under a CC BY-NC-SA 3.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

1.8: Improper Functions (Exercises)

[exer:1] Suppose g and h are differentiable on $[a,b]$, with

$$a \le g(y) \le b \quad \text{and} \quad a \le h(y) \le b, \quad c \le y \le d. \tag{1.8.1}$$

Let f and f_y be continuous on $[a,b]\times[c,d]$. Derive *Liebniz's rule*:

$$\begin{aligned}\frac{d}{dy}\int_{g(y)}^{h(y)} f(x,y)\,dx = \quad & f(h(y),y)h'(y) - f(g(y),y)g'(y) \\ & + \int_{g(y)}^{h(y)} f_y(x,y)\,dx.\end{aligned} \tag{1.8.2}$$

(Hint: Define $H(y,u,v) = \int_u^v f(x,y)\,dx$ and use the chain rule.)

[exer:2] Adapt the proof of Theorem [theorem:2] to prove Theorem [theorem:3].

[exer:3] Adapt the proof of Theorem [theorem:4] to prove Theorem [theorem:5].

[exer:4] Show that Definition [definition:3] is independent of c; that is, if $\int_a^c f(x,y)\,dx$ and $\int_c^b f(x,y)\,dx$ both converge uniformly on S for some $c\in(a,b)$, then they both converge uniformly on S and every $c\in(a,b)$.

[exer:5]

()0pt0pt14pt 8pt22pt0pt

Show that if f is bounded on $[a,b]\times[c,d]$ and $\int_a^b f(x,y)\,dx$ exists as a proper integral for each $y\in[c,d]$, then it converges uniformly on $[c,d]$ according to all of Definition [definition:1]–[definition:3].

Give an example to show that the boundedness of f is essential in **(a)**.

[exer:6] Working directly from Definition [definition:1], discuss uniform convergence of the following integrals:

(a) $\int_0^\infty \dfrac{dx}{1+y^2x^2}\,dx$	**(b)** $\int_0^\infty e^{-xy}x^2\,dx$
(c) $\int_0^\infty x^{2n}e^{-yx^2}\,dx$	**(d)** $\int_0^\infty \sin xy^2\,dx$
(e) $\int_0^\infty (3y^2-2xy)e^{-y^2x}\,dx$	**(f)** $\int_0^\infty (2xy-y^2x^2)e^{-xy}\,dx$

[exer:7] Adapt the proof of Theorem [theorem:6] to prove Theorem [theorem:7].

[exer:8] Use Weierstrass's test to show that the integral converges uniformly on S :

()0pt0pt14pt 8pt22pt0pt

$\int_0^\infty e^{-xy}\sin x\,dx$, $S=[\rho,\infty)$,$\rho>0$

$\int_0^\infty \dfrac{\sin x}{x^y}\,dx$, $S=[c,d]$, $1<c<d<2$

$\int_1^\infty e^{-px}\dfrac{\sin xy}{x}\,dx$, $p>0$, $S=(-\infty,\infty)$

$\int_0^1 \dfrac{e^{xy}}{(1-x)^y}\,dx$, $S=(-\infty,b)$,$b<1$

$\int_{-\infty}^\infty \dfrac{\cos xy}{1+x^2y^2}\,dx$, $S=(-\infty,-\rho]\cup[\rho,\infty)$,$\rho>0$.

$\int_1^\infty e^{-x/y}\,dx$, $S=[\rho,\infty)$,$\rho>0$

$\int_{-\infty}^\infty e^{xy}e^{-x^2}\,dx$, $S=[-\rho,\rho]$,$\rho>0$

$\int_0^\infty \frac{\cos xy - \cos ax}{x^2}\,dx$, $S = (-\infty, \infty)$

$\int_0^\infty x^{2n} e^{-yx^2}\,dx$, $S = [\rho, \infty)$,$\rho > 0$, $n = 0, 1, 2,\ldots$

[exer:9]

()0pt0pt14pt 8pt22pt0pt

Show that

$$\Gamma(y) = \int_0^\infty x^{y-1} e^{-x}\,dx \tag{1.8.3}$$

converges if $y > 0$, and uniformly on $[c, d]$ if $0 < c < d < \infty$.

Use integration by parts to show that

$$\Gamma(y) = \frac{\Gamma(y+1)}{y}, \quad y \geq 0, \tag{1.8.4}$$

and then show by induction that

$$\Gamma(y) = \frac{\Gamma(y+n)}{y(y+1)\cdots(y+n-1)}, \quad y > 0, \quad n = 1, 2, 3, \ldots. \tag{1.8.5}$$

How can this be used to define $\Gamma(y)$ in a natural way for all $y \neq 0, -1, -2, \ldots$? (This function is called the *gamma function*.)

Show that $\Gamma(n+1) = n!$ if n is a positive integer.

Show that

$$\int_0^\infty e^{-st} t^\alpha\,dt = s^{-\alpha-1}\Gamma(\alpha+1), \quad \alpha > -1, \quad s > 0. \tag{1.8.6}$$

[exer:10] Show that Theorem [theorem:8] remains valid with assumption **(c)** replaced by the assumption that $|g_x(x, y)|$ is monotonic with respect to x for all $y \in S$.

[exer:11] Adapt the proof of Theorem [theorem:8] to prove Theorem [theorem:9].

[exer:12] Use Dirichlet's test to show that the following integrals converge uniformly on $S = [\rho, \infty)$ if $\rho > 0$:

(a) $\int_1^\infty \frac{\sin xy}{x^y}\,dx$	**(b)** $\int_2^\infty \frac{\sin xy}{\log x}\,dx$
(c) $\int_0^\infty \frac{\cos xy}{x + y^2}\,dx$	**(d)** $\int_1^\infty \frac{\sin xy}{1 + xy}\,dx$

[exer:13] Suppose g, g_x and h are continuous on $[a, b) \times S$, and denote $H(x, y) = \int_a^x h(u, y)\,du$, $a \leq x < b$. Suppose also that

$$\lim_{x\to b-} \left\{ \sup_{y\in S} |g(x, y)H(x, y)| \right\} = 0 \quad \text{and} \quad \int_a^b g_x(x, y)H(x, y)\,dx \tag{1.8.7}$$

converges uniformly on S. Show that $\int_a^b g(x, y)h(x, y)\,dx$ converges uniformly on S.

[exer:14] Prove Theorem [theorem:10] for the case where $f = f(x, y)$ is continuous on $(a, b] \times [c, d]$.

[exer:15] Prove Theorem [theorem:11] for the case where $f = f(x, y)$ is continuous on $(a, b] \times [c, d]$.

[exer:16] Show that

$$C(y) = \int_{-\infty}^\infty f(x)\cos xy\,dx \quad \text{and} \quad S(y) = \int_{-\infty}^\infty f(x)\sin xy\,dx \tag{1.8.8}$$

are continuous on $(-\infty, \infty)$ if

$$\int_{-\infty}^\infty |f(x)|\,dx < \infty. \tag{1.8.9}$$

$-\infty$

[exer:17] Suppose f is continuously differentiable on $[a,\infty)$, $\lim_{x\to\infty} f(x)=0$, and

$$\int_a^\infty |f'(x)|\,dx<\infty. \tag{1.8.10}$$

Show that the functions

$$C(y)=\int_a^\infty f(x)\cos xy\,dx \quad\text{and}\quad S(y)=\int_a^\infty f(x)\sin xy\,dx \tag{1.8.11}$$

are continuous for all $y\neq 0$. Give an example showing that they need not be continuous at $y=0$.

[exer:18] Evaluate $F(y)$ and use Theorem [theorem:11] to evaluate I:

()0pt0pt14pt 8pt22pt0pt

$F(y)=\displaystyle\int_0^\infty \frac{dx}{1+y^2x^2}$, $y\neq 0$; $I=\displaystyle\int_0^\infty \frac{\tan^{-1}ax-\tan^{-1}bx}{x}\,dx$, $a,\ b>0$

$F(y)=\displaystyle\int_0^\infty x^y\,dx$, $y>-1$; $I=\displaystyle\int_0^\infty \frac{x^a-x^b}{\log x}\,dx$, $a,\ b>-1$

$F(y)=\displaystyle\int_0^\infty e^{-xy}\cos x\,dx$, $y>0$

$I=\displaystyle\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\cos x\,dx$, $a,\ b>0$

$F(y)=\displaystyle\int_0^\infty e^{-xy}\sin x\,dx$, $y>0$

$I=\displaystyle\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\sin x\,dx$, $a,\ b>0$

$F(y)=\displaystyle\int_0^\infty e^{-x}\sin xy\,dx$; $I=\displaystyle\int_0^\infty e^{-x}\frac{1-\cos ax}{x}\,dx$

$F(y)=\displaystyle\int_0^\infty e^{-x}\cos xy\,dx$; $I=\displaystyle\int_0^\infty e^{-x}\frac{\sin ax}{x}\,dx$

[exer:19] Use Theorem [theorem:11] to evaluate:

()0pt0pt14pt 8pt22pt0pt

$\displaystyle\int_0^1 (\log x)^n x^y\,dx$, $y>-1$,$n=0,\ 1,\ 2,\ldots$.

$\displaystyle\int_0^\infty \frac{dx}{(x^2+y)^{n+1}}\,dx$, $y>0$, $n=0,\ 1,\ 2,\ \ldots$.

$\displaystyle\int_0^\infty x^{2n+1}e^{-yx^2}\,dx$, $y>0$, $n=0,\ 1,\ 2,\ldots$.

$\displaystyle\int_0^\infty xy^x\,dx$, $0<y<1$.

[exer:20]

()0pt0pt14pt 8pt22pt0pt

Use Theorem [theorem:11] and integration by parts to show that

$$F(y)=\int_0^\infty e^{-x^2}\cos 2xy\,dx \tag{1.8.12}$$

satisfies

$$F'+2yF=0. \tag{1.8.13}$$

Use part **(a)** to show that

$$F(y)=\frac{\sqrt{\pi}}{2}e^{-y^2}. \tag{1.8.14}$$

[exer:21] Show that

$$\int_0^\infty e^{-x^2}\sin 2xy\,dx = e^{-y^2}\int_0^y e^{u^2}\,du. \tag{1.8.15}$$

(Hint: See Exercise [exer:20].)

[exer:22] State a condition implying that

$$C(y)=\int_a^\infty f(x)\cos xy\,dx \quad\text{and}\quad S(y)=\int_a^\infty f(x)\sin xy\,dx \tag{1.8.16}$$

are n times differentiable on for all $y\neq 0$. (Your condition should imply the hypotheses of Exercise [exer:16].)

[exer:23] Suppose f is continuously differentiable on $[a,\infty)$,

$$\int_a^\infty |(x^k f(x))'|\,dx<\infty, \quad 0\le k\le n, \tag{1.8.17}$$

and $\lim_{x\to\infty} x^n f(x)=0$. Show that if

$$C(y)=\int_a^\infty f(x)\cos xy\,dx \quad\text{and}\quad S(y)=\int_a^\infty f(x)\sin xy\,dx, \tag{1.8.18}$$

then

$$C^{(k)}(y)=\int_a^\infty x^k f(x)\cos xy\,dx \quad\text{and}\quad S^{(k)}(y)=\int_a^\infty x^k f(x)\sin xy\,dx, \tag{1.8.19}$$

$0\le k\le n$.

[exer:24] Differentiating

$$F(y)=\int_1^\infty \cos\frac{y}{x}\,dx \tag{1.8.20}$$

under the integral sign yields

$$-\int_1^\infty \frac{1}{x}\sin\frac{y}{x}\,dx, \tag{1.8.21}$$

which converges uniformly on any finite interval. (Why?) Does this imply that F is differentiable for all y?

[exer:25] Show that Theorem [theorem:11] and induction imply Eq. [eq:30].

[exer:26] Prove Theorem [theorem:12].

[exer:27] Show that if $F(s)=\int_0^\infty e^{-sx}f(x)\,dx$ converges for $s=s_0$, then it converges uniformly on $[s_0,\infty)$. (What's the difference between this and Theorem [theorem:13]?)

[exer:28] Prove: If f is continuous on $[0,\infty)$ and $\int_0^\infty e^{-s_0x}f(x)\,dx$ converges, then

$$\lim_{s\to s_0+}\int_0^\infty e^{-sx}f(x)\,dx=\int_0^\infty e^{-s_0x}f(x)\,dx. \tag{1.8.22}$$

(Hint: See the proof of Theorem 4.5.12, p. 273.)

[exer:29] Under the assumptions of Exercise [exer:28], show that

$$\lim_{s\to s_0+}\int_r^\infty e^{-sx}f(x)\,dx=\int_r^\infty e^{-s_0x}f(x)\,dx, \quad r>0. \tag{1.8.23}$$

[exer:30] Suppose f is continuous on $[0,\infty)$ and

$$F(s)=\int_0^\infty e^{-sx}f(x)\,dx \tag{1.8.24}$$

converges for $s=s_0$. Show that $\lim_{s\to\infty}F(s)=0$. (Hint: Integrate by parts.)

[exer:31]

()0pt0pt14pt 8pt22pt0pt

Starting from the result of Exercise [exer:18]**(d)**, let $b \to \infty$ and invoke Exercise [exer:30] to evaluate

$$\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx, \quad a > 0. \tag{1.8.25}$$

Use **(a)** and Exercise [exer:28] to show that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \tag{1.8.26}$$

[exer:32]

()0pt0pt14pt 8pt22pt0pt

Suppose f is continuously differentiable on $[0, \infty)$ and

$$|f(x)| \le Me^{s_0 x}, \quad 0 \le x \le \infty. \tag{1.8.27}$$

Show that

$$G(s) = \int_0^\infty e^{-sx} f'(x)\, dx \tag{1.8.28}$$

converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$. (Hint: Integrate by parts.)

Show from part **(a)** that

$$G(s) = \int_0^\infty e^{-sx} x e^{x^2} \sin e^{x^2}\, dx \tag{1.8.29}$$

converges uniformly on $[\rho, \infty)$ if $\rho > 0$. (Notice that this does not follow from Theorem [theorem:6] or [theorem:8].)

[exer:33] Suppose f is continuous on $[0, \infty)$,

$$\lim_{x \to 0+} \frac{f(x)}{x} \tag{1.8.30}$$

exists, and

$$F(s) = \int_0^\infty e^{-sx} f(x)\, dx \tag{1.8.31}$$

converges for $s = s_0$. Show that

$$\int_{s_0}^\infty F(u)\, du = \int_0^\infty e^{-s_0 x} \frac{f(x)}{x}\, dx. \tag{1.8.32}$$

Answers to selected exercises

If $f(x, y) = 1/y$ for $y \ne 0$ and $f(x, 0) = 1$, then $\int_a^b f(x, y)\, dx$ does not converge uniformly on $[0, d]$ for any $d > 0$.

, **(d)**, and **(e)** converge uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$; **(b)**, **(c)**, and **(f)** converge uniformly on $[\rho, \infty)$ if $\rho > 0$.

Let $C(y) = \int_1^\infty \frac{\cos xy}{x}\, dx$ and $S(y) = \int_1^\infty \frac{\sin xy}{x}\, dx$. Then $C(0) = \infty$ and $S(0) = 0$, while $S(y) = \pi/2$ if $y \ne 0$.

$F(y) = \dfrac{\pi}{2|y|}; I = \dfrac{\pi}{2} \log \dfrac{a}{b}$ $F(y) = \dfrac{1}{y+1}; I = \log \dfrac{a+1}{b+1}$

$F(y) = \dfrac{y}{y^2+1}$; $I = \dfrac{1}{2} \dfrac{b^2+1}{a^2+1}$

(d) $F(y) = \dfrac{1}{y^2+1}; I = \tan^{-1} b - \tan^{-1} a$

(e) $F(y) = \dfrac{y}{y^2+1}; I = \dfrac{1}{2} \log(1 + a^2)$

(f) $F(y) = \dfrac{1}{y^2+1}; I = \tan^{-1} a$

$(-1)^n n!(y+1)^{-n-1}\ \pi 2^{-2n-1}\binom{2n}{n} y^{-n-1/2}$

(c) $\dfrac{n!}{2y^{n+1}}\ (\log y)^{-2}$ **(d)** $\dfrac{1}{(\log x)^2}$

$\displaystyle\int_{-\infty}^{\infty} |x^n f(x)|\,dx < \infty$

No; the integral defining F diverges for all y.

$\dfrac{\pi}{2} - \tan^{-1} a$

Beginning of manual

1. If $H(y,u,v) = \displaystyle\int_u^v f(x,y)\,dx$ then

$$H_u(y,u,v) = -f(u,y), \quad H_v(y,u,v) = f(v,y), \tag{1.8.33}$$

and, by Theorem 1, $H_y(u,v,y) = \displaystyle\int_u^v f_y(x,y)\,dx$. If

$$F(y) = H(y,g(y),h(y)) = \int_{g(y)}^{h(y)} f(x,y)\,dx, \tag{1.8.34}$$

then

$$\begin{aligned} F'(y) = &\quad H_v(y,g(y),h(y))h'(y) + H_u(y,g(y),h(y))g'(y) + H_y(y,g(y),h(y)) \\ = &\quad f(h(y),y)h'(y) - f(g(y),y)g'(y) + \int_{g(y)}^{h(y)} f_y(x,y)\,dx. \end{aligned} \tag{1.8.35}$$

2. Theorem 3 (Cauchy Criterion for Convergence of an Improper Integral II) *Suppose g is integrable on every finite closed subinterval of $(a,b]$ and denote*

$$G(r) = \int_r^b g(x)\,dx, \quad a < r \le b. \tag{1.8.36}$$

Then the improper integral $\int_a^b g(x)\,dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0 \in (a,b]$ such that

$$|G(r) - G(r_1)| \le \epsilon, \quad a < r, r_1 \le r_0. \tag{A}$$

For necessity, suppose $\int_a^b g(x)\,dx = L$. By definition, this means that for each $\epsilon > 0$ there is an $r_0 \in (a,b]$ such that

$$|G(r) - L| < \frac{\epsilon}{2} \quad \text{and} \quad |G(r_1) - L| < \frac{\epsilon}{2}, \quad a < r, r_1 \le r_0. \tag{1.8.37}$$

Therefore,

$$\begin{aligned} |G(r) - G(r_1)| = &\quad |(G(r) - L) - (G(r_1) - L)| \\ \le &\quad |G(r) - L| + |G(r_1) - L| \le \epsilon, \quad a < r, r_1 \le r_0. \end{aligned} \tag{1.8.38}$$

For sufficiency, (A) implies that

$$|G(r)| = |G(r_1) + (G(r) - G(r_1))| \le |G(r_1)| + |G(r) - G(r_1)| \le |G(r_1)| + \epsilon, \tag{1.8.39}$$

$a < r_1 \le r_0$. Since G is also bounded on the compact set $[r_0, b]$ (Theorem 5.2.11, p. 313), G is bounded on $(a,b]$. Therefore the monotonic functions

$$\overline{G}(r) = \sup\{G(r_1) \,|\, a < r_1 \le r\} \quad \text{and} \quad \underline{G}(r) = \inf\{G(r_1) \,|\, a < r_1 \le r\} \tag{1.8.40}$$

are well defined on $(a,b]$, and

$$\lim_{r\to a+} \overline{G}(r) = \overline{L} \quad \text{and} \quad \lim_{r\to a+} \underline{G}(r) = \underline{L} \tag{1.8.41}$$

both exist and are finite (Theorem 2.1.11, p. 47). From (A),

$$\begin{aligned} |G(r) - G(r_1)| = &\quad |(G(r) - G(r_0)) - (G(r_1) - G(r_0))| \\ \le &\quad |G(r) - G(r_0)| + |G(r_1) - G(r_0)| \le 2\epsilon, \end{aligned} \tag{1.8.42}$$

so $\overline{G}(r) - \underline{G}(r) \le 2\epsilon$. Since ϵ is an arbitrary positive number, this implies that

$$\lim_{r\to a+} (\overline{G}(r) - \underline{G}(r)) = 0, \tag{1.8.43}$$

so $\overline{L} = \underline{L}$. Let $L = \overline{L} = \underline{L}$. Since

$$\underline{G}(r) \le G(r) \le \overline{G}(r), \tag{1.8.44}$$

it follows that $\lim_{r\to a+} G(r) = L$.

3. Theorem 5 (Cauchy Criterion for Uniform Convergence II) *The improper integral*

$$\int_a^b f(x,y)\,dx = \lim_{r\to a+} \int_r^b f(x,y)\,dx \tag{1.8.45}$$

converges uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0 \in (a,b]$ such that

$$\left|\int_{r_1}^{r} f(x,y)\,dx\right| < \epsilon, \quad y \in S, \quad a < r, r_1 \le r_0. \tag{A}$$

Suppose $\int_a^b f(x,y)\,dx$ converges uniformly on S and $\epsilon > 0$. From Definition 2, there is an $r_0 \in (a,b]$ such that

$$\left|\int_a^r f(x,y)\,dx\right| < \frac{\epsilon}{2} \quad \text{and} \quad \left|\int_a^{r_1} f(x,y)\,dx\right| < \frac{\epsilon}{2},\ y \in S,\ a < r, r_1 \le r_0. \tag{B}$$

Since

$$\int_{r_1}^{r} f(x,y)\,dx = \int_{r_1}^{b} f(x,y)\,dx - \int_r^b f(x,y)\,dx \tag{1.8.46}$$

(B) and the triangle inequality imply (A).

For the converse, denote

$$F(y) = \int_r^b f(x,y)\,dx. \tag{1.8.47}$$

Since (A) implies that

$$|F(r,y) - F(r_1,y)| \le \epsilon, \quad y \in S, \quad a < r, r_1 \le r_0, \tag{1.8.48}$$

Theorem 2 with $G(r) = F(r,y)$ (y fixed but arbitrary in S) implies that $\int_a^b f(x,y)\,dx$ converges pointwise for $y \in S$. Therefore, if $\epsilon > 0$ then, for each $y \in S$, there is an $r_0(y) \in (a,b]$ such that

$$\left|\int_a^r f(x,y)\,dx\right| \le \epsilon, \quad y \in S, \quad a < r \le r_0(y). \tag{C}$$

For each $y \in S$, choose $r_1(y) \le \min[r_0(y), r_0]$. Then

$$\int_a^r f(x,y)\,dx = \int_a^{r_1(y)} f(x,y)\,dx + \int_{r_1(y)}^r f(x,y)\,dx, \tag{1.8.49}$$

so (A), (C), and the triangle inequality imply that

$$\left|\int_a^r f(x,y)\,dx\right| \le 2\epsilon, \quad y \in S, \quad a < r \le r_0 \tag{1.8.50}$$

4. From Definition 3, $\int_a^b f(x,y)\,dx$ converges uniformly on S if and only if $\int_a^c f(x,y)\,dx$ and $\int_c^b f(x,y)\,dx$ both converge uniformly on S, where $c \in (a,b)$. From Theorems 4 and Theorem 5, this is true if and only if, for any $\epsilon > 0$ there are points r_0 and s_0 in (a,b) such that

$$\left|\int_r^{r_1} f(x,y)\,dx\right| \le \epsilon, \quad y \in S, \quad r_0 \le r, r_1 < b \tag{1.8.51}$$

and

$$\left|\int_r^{s} f(x,y)\,dx\right| < \epsilon, \quad y \in S, \quad a < r, s \le s_0 \tag{1.8.52}$$

https://math.libretexts.org/@go/page/17329

$$\left|\int_{s_1} f(x,y)\,dx\right| < \epsilon, \quad y \in S, \quad a < s, s_1 < s_0. \tag{1.8.52}$$

These conditions are independent of c.

5. (a) If $|f(x,y)| \le M$ on $[a,b]\times[c,d]$ then

$$\left|\int_{r_1}^{r_2} f(x,y)\,dx\right| \le M|r_2 - r_1| \tag{1.8.53}$$

so the Cauchy convergence theorems imply the conclusion.

Define $f = f(x,y)$ on $[0,1]\times[0,1]$ by

$$f(x,y) = \begin{cases} \dfrac{1}{y} & \text{if} \quad 0 < y \le 1, \\ 1 & \text{if} \quad y = 0. \end{cases} \tag{1.8.54}$$

Then

$$\int_{r_1}^{r_2} f(x,y)\,dx = \begin{cases} \dfrac{r_2 - r_1}{y} & \text{if} \quad 0 < y \le 1, \\ r_2 - r_1 & \text{if} \quad y = 0. \end{cases} \tag{1.8.55}$$

Therefore f does not satisfy the requirements of Cauchy's convergence theorems.

6. In all parts $I(y)$ denotes the given integral.

$I(0) = \infty$. If $y \ne 0$ let $u = xy$; then $I(y) = \dfrac{1}{y}\displaystyle\int_0^\infty \frac{du}{1+u^2}$. If $\rho > 0$ and $\epsilon > 0$, choose r so that $\displaystyle\int_r^\infty \frac{du}{1+u^2} < \rho\epsilon$. Then $\dfrac{1}{|y|}\displaystyle\int_r^\infty \frac{du}{1+u^2} < \epsilon$ if $|y| \ge \rho$, so $I(y)$ converges uniformly on $(-\infty,\rho]\bigcup[\rho,\infty)$ if $\rho > 0$.

$I(y) = \infty$ if $y \le 0$. If $y > 0$ let $u = xy$; then $I(y) = \dfrac{1}{y^3}\displaystyle\int_0^\infty e^{-u}u^2\,du$. If $\rho > 0$ and $\epsilon > 0$, choose r so that $\displaystyle\int_r^\infty e^{-u}u^2\,du < \rho^3\epsilon$. Then $\dfrac{1}{y^3}\displaystyle\int_r^\infty e^{-u}u^2\,du < \epsilon$ if $y \ge \rho$, so $I(y)$ converges uniformly on $[\rho,\infty)$ if $\rho > 0$.

$I(y) = \infty$ if $y \le 0$. If $y > 0$ let $u = xy^{1/2}$; then $I(y) = y^{-n-1/2}\displaystyle\int_0^\infty u^{2n}e^{-u}\,du$. If $\rho > 0$ and $\epsilon > 0$, we can choose r so that $\displaystyle\int_r^\infty u^{2n}e^{-u}\,du < \epsilon\rho^{n+1/2}$. Then $y^{-n-1/2}\displaystyle\int_r^\infty u^{2n}e^{-u}\,du < \epsilon$ if $y \ge \rho$, so $I(y)$ converges uniformly on $S = [\rho,\infty)$ if $\rho > 0$.

Since $I(-y) = -I(y)$, it suffices to assume that $y > 0$. If $u = yx^2$ then $I(y) = \dfrac{1}{2\sqrt{y}}\displaystyle\int_0^\infty \frac{\sin u\,du}{\sqrt{u}}$. From Example 3.4.14 (p. 162), this integral converges conditionally. If $\rho > 0$ and $\epsilon > 0$, we can choose r so that $\left|\displaystyle\int_r^\infty \frac{\sin u\,du}{\sqrt{u}}\right| < 2\epsilon\sqrt{\rho}$, so $I(y)$ converges uniformly on $(-\infty,-\rho]\bigcup[\rho,\infty)$ if $\rho > 0$.

If $u = y^2x$ then $I(y) = 3\displaystyle\int_0^\infty e^{-u}\,du - \frac{2}{y^3}\int_0^\infty ue^{-u}\,du$. If $\rho > 0$, we can choose r so that $3\displaystyle\int_r^\infty e^{-u}\,du < \frac{\epsilon}{2}$ and $\displaystyle\int_r^\rho ue^{-u}\,du < \frac{\rho^3\epsilon}{2}$. Then

$$\left|3\int_r^\infty e^{-u}\,du - \frac{3}{y^3}\int_r^\infty ue^{-u}\,du\right| < \epsilon \quad \text{if} \quad |y| \ge \rho, \tag{1.8.56}$$

so $I(y)$ converges uniformly on $(-\infty,\rho]\bigcup[\rho,\infty)$ if $\rho > 0$.

$I(y) = -\infty$ if $y \le 0$. If $y > 0$, let $u = xy$; then $I(y) = \dfrac{1}{y}\displaystyle\int_0^\infty (2u - u^2)e^{-u}\,du$. If $\rho > 0$ and $\epsilon > 0$, we can choose r so that $\displaystyle\int_r^\infty |2u - u^2|e^{-u}\,du < \epsilon\rho$, so $I(y)$ converges uniformly on $S = [\rho,\infty)$ if $\rho > 0$.

7. Theorem 7 (Weierstrass's Test for Absolute Uniform Convergence II) Suppose $f = f(x,y)$ is locally integrable $(a,b]$ and, for some $b_0 \in (a,b]$,

$$|f(x,y)| \le M(x),\ y \in S,\ x \in (a,b_0], \tag{A}$$

where

$$\int_a^{b_0} M(x)\,dx < \infty. \tag{1.8.57}$$

Then $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S.

Denote $\int_a^{b_0} M(x)\,dx = L < \infty$. By definition, for each $\epsilon > 0$ there is an $r_0 \in (a, b_0]$ such that

$$L - \epsilon \le \int_r^{b_0} M(x)\,dx \le L, \quad a < r \le r_0. \tag{1.8.58}$$

Therefore, if $a < r_1 < r \le r_0$, then

$$0 \le \int_{r_1}^{r} M(x)\,dx = \left(\int_{r_1}^{b_0} M(x)\,dx - L\right) - \left(\int_{r}^{b_0} M(x)\,dx - L\right) < \epsilon. \tag{1.8.59}$$

This and (A) imply that

$$\int_{r_1}^{r} |f(x,y)|\,dx \le \int_{r_1}^{r} M(x)\,dx < \epsilon,\ y \in S,\ a < r_1 \le r_0 \le b. \tag{1.8.60}$$

Now Theorem 5 implies the stated conclusion.

$|e^{-xy}\sin x| \le e^{-\rho x}$ if $y \ge \rho$ and $\int_\rho^\infty e^{-\rho x}\,dx < \infty$.

$\int_0^\infty \dfrac{\sin x\,dx}{x^y} = I_1(y) + I_2(y)$, where

$$I_1(y) = \int_0^1 \frac{\sin x\,dx}{x^y} \quad \text{and} \quad I_2(y) = \int_1^\infty \frac{\sin x\,dx}{x^y} \tag{1.8.61}$$

are both improper integrals. Since

$$\sin x = x - \left(\frac{x^3}{3!} - \frac{x^5}{5!}\right) - \left(\frac{x^7}{7!} - \frac{x^9}{9!}\right) + \cdots < x, \quad 0 \le x \le 1. \tag{1.8.62}$$

If $0 \le 1$ and $y \le d \le 2$, then

$$\left|\frac{\sin x}{x^y}\right| \le x^{1-y} \le x^{1-d} \text{ so } \int_0^1 x^{-1+y}\,dx < \int_0^1 x^{-1+d}\,dx = \frac{1}{2-d} \tag{1.8.63}$$

so $I_1(y)$ converges absolutely uniformly on S. Since $c > 1$,

$$\frac{|\sin x|}{x^y} \le x^{-c} \quad \text{and} \quad \int_1^\infty x^{-c}\,dx = \frac{1}{c-1} \quad \text{if} \tag{1.8.64}$$

$I_2(y)$ converges absolutely uniformly on S.

If $x \ge 1$ then $e^{-px}\left|\dfrac{\sin xy}{x}\right| \le e^{-px}$ for all y and $\int_1^\infty e^{-px}\,dx < \infty$, since $p > 0$.

$\dfrac{e^{xy}}{(1-x)^y} \le \dfrac{e^b}{(1-x)^b}$, if $0 \le x < 1$ and $y \le b$, and $\int_0^1 (1-x)^{-b}\,dx < \infty$ if $b < 1$.

If $|y| \ge \rho > 0$ then $\left|\dfrac{\cos xy}{1+x^2y^2}\right| \le \dfrac{1}{1+\rho^2x^2}$ for all x, and $\int_0^\infty \dfrac{dx}{1+\rho^2x^2} < \infty$.

If $y \ge \rho > 0$ then $e^{-x/y} \le e^{-x/\rho}$ and $\int_0^\infty e^{-x/\rho}\,dx < \infty$.

If $|y| \le \rho$ then $e^{xy}e^{-x^2} \le e^{x\rho}e^{-x^2}$ and $\int_{-\infty}^\infty e^{x\rho}e^{-x^2}\,dx < \infty$.

If $|x| \ge 1$ then $|\cos xy - \cos ax| \le 2$ and $\int_1^\infty \dfrac{2\,dx}{x^2} < \infty$.

If $y \ge \rho > 0$ then $e^{-yx^2} \le e^{-\rho x^2}$ and $\int_0^\infty x^{2n}e^{-\rho x^2}\,dx < \infty$.

9. (a) If $0 < x < 1$ then $|x^{y-1}e^{-x}| < x^{c-1}$. Therefore, since $\int_0^1 x^{c-1}\,dx < \infty$ if $c > 0$, $\int_0^1 x^{y-1}e^{-x}\,dx$ converges uniformly on $[c,\infty)$ if $c > 0$, by Theorem 7. If $x > 1$ then $|x^{y-1}e^{-x}| \le x^{d-1}e^{-x}$ if $y \le d$. Therefore, since $\int_1^\infty x^{d-1}e^{-x}\,dx < \infty$, $\int_1^\infty x^{y-1}e^{-x}\,dx$ converges uniformly on $(-\infty, d]$ for every d, by Theorem 6. Hence, $\int_0^\infty x^{y-1}e^{-x}\,dx$ converges uniformly on $[c,d]$ if $c > 0$.

If $y > 0$ then

$$\Gamma(y) = \int_0^\infty x^{y-1}e^{-x}\,dx = \frac{x^y e^{-x}}{y}\bigg|_0^\infty + \frac{1}{y}\int_0^\infty x^y e^{-x}\,dx = \frac{\Gamma(y+1)}{y}. \tag{A}$$

Therefore

$$\Gamma(y) = \frac{\Gamma(y+n)}{y(y+1)\cdots(y+n-1)} \tag{B}$$

is true when $n = 1$. Now suppose it is true for given positive integer n, and replace y by $y + n$ in (A):

$$\Gamma(y+n) = \frac{\Gamma(y+n+1)}{y+n}. \tag{1.8.65}$$

Substituting this into (B) yields

$$\Gamma(y) = \frac{\Gamma(y+n+1)}{y(y+1)\cdots(y+n)}, \tag{1.8.66}$$

which completes the induction.

If $-n < y < -n+1$ with $n \le 1$ then $0 < y+n < 1$ and we can compute $\Gamma(y+n)$ from the definition in part **(a)**:

$$\Gamma(y+n) = \int_0^\infty x^{y+n-1}e^{-x}\,dx. \tag{1.8.67}$$

Then we can define $\Gamma(y)$ by (B).

The assertion is true if $n = 1$, since

$$\Gamma(2) = \int_0^\infty xe^{-x}\,dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx = 1. \tag{1.8.68}$$

If $\Gamma(n+1) = n!$ for some $n \ge 1$, then

$$\begin{aligned}\Gamma(n+2) = &\int_0^\infty x^{n+1}e^{-x}\,dx = -e^{-x}x^{n+1}\Big|_0^\infty + (n+1)\int_0^\infty x^{n+1}e^{-x}\,dx\\ = &\quad (n+1)\Gamma(n+1) = (n+1)n! = (n+1)!,\end{aligned} \tag{1.8.69}$$

which completes the induction proof.

The change of variable $x = st$ yields

$$\int_0^\infty e^{-st}t^\alpha\,dt = \frac{1}{s^{\alpha+1}}\int_0^\infty x^\alpha e^{-x}\,dx = \frac{1}{s^{\alpha+1}}\Gamma(\alpha+1), \tag{1.8.70}$$

from the definition of the Gamma function.

Since $|g_x(x,y)|$ is monotonic with respect to x,

$$\int_r^{r_1} |g_x(x,y)|\,dx = |g(r_1,y) - g(r,y)|, \quad a \le r < r_1 < b. \tag{A}$$

From Assumption **(a)** of Theorem 8, if $\epsilon > 0$ there is an $r_0 \in [a,b)$ such that

$$|g(s,y)| \le \epsilon, \quad y \in S, \quad r_0 \le s < b. \tag{1.8.71}$$

Therefore, (A) implies that

$$\int_r^{r_1} |g_x(x,y)\,dx \le 2\epsilon, \quad y \in S, \quad r_0 \le r \le r_1 < b. \tag{1.8.72}$$

Now Theorem 4 implies that $\int_a^b |g(x,y)|\,dx$ converges uniformly on S, which is assumption **(c)** of Theorem 8.

) If g, g_x, and h are continuous on $(a,b] \times S$ then

$$\int_a^b g(x,y)h(x,y)\,dx \tag{1.8.73}$$

converges uniformly on S if the following conditions are satisfied:

()0pt0pt14pt 8pt22pt0pt

$$\lim_{x\to a+}\left\{\sup_{y\in S}|g(x,y)|\right\}=0;$$

There is a constant M such that

$$\sup_{y\in S}\left|\int_x^b h(u,y)\,du\right|\le M,\quad a<x\le b; \tag{1.8.74}$$

$\int_a^b |g_x(x,y)|\,dx$ converges uniformly on S.

If

$$H(x,y)=\int_x^b h(u,y)\,du \tag{A}$$

then integration by parts yields

$$\begin{aligned}\int_{r_1}^r g(x,y)h(x,y)\,dx &= -\int_{r_1}^r g(x,y)H_x(x,y)\,dx\\ &= -g(r,y)H(r,y)+g(r_1,y)H(r_1,y)\\ &\quad +\int_{r_1}^r g_x(x,y)H(x,y)\,dx.\end{aligned} \tag{1.8.75}$$

Therefore, since assumption **(b)** and (A) imply that $|H(x,y)|\le M$, $(x,y)\in(a,b]\times S$,

$$\left|\int_{r_1}^r g(x,y)h(x,y)\,dx\right|\le M\left(2\sup_{a<x\le r}|g(x,y)|+\int_{r_1}^r|g_x(x,y)|\,dx\right) \tag{B}$$

on $[r_1,r]\times S$. Now suppose $\epsilon>0$. From assumption **(a)**, there is an $r_0\in[a,b)$ such that $|g(x,y)|<\epsilon$ on S if $a<x\le r_0\le b$. From assumption **(c)** and Theorem 5, there is an $s_0\in(a,b]$ such that

$$\int_{r_1}^r|g_x(x,y)|\,dx<\epsilon,\quad y\in S,\quad a<r_1<r\le s_0. \tag{1.8.76}$$

Therefore (B) implies that

$$\left|\int_{r_1}^r g(x,y)h(x,y)\right|<3M\epsilon,\quad y\in S,\quad a<r_1<r\min(r_0,s_0) \tag{1.8.77}$$

Now Theorem 5 implies the stated conclusion.

Denote $F(y)=\int_1^\infty \dfrac{\sin xy}{x^y}$ and, with $1\le r<r_1$,

$$\begin{aligned}F(r,r_1,y)=\int_r^{r_1}\frac{\sin xy}{x^y}\,dx &= -\frac{\cos xy}{yx^y}\Big|_r^{r_1}-\int_r^{r_1}\frac{\cos xy}{x^{y+1}}\,dx\\ &= \frac{\cos ry}{yr^y}-\frac{\cos r_1y}{yr_1^y}-\int_r^{r_1}\frac{\cos xy}{x^{y+1}}\,dx.\end{aligned} \tag{1.8.78}$$

Therefore

$$|F(r,r_1,y)|\le\frac{2}{yr^y}+\int_r^{r_1}x^{-y-1}\,dx<\frac{3}{yr^y},\quad r,y>0. \tag{1.8.79}$$

Now Theorem 4 implies that $F(y)$ converges uniformly on $[\rho,\infty)$ if $\rho>0$.

(b) Denote $F(y)=\int_2^\infty \dfrac{\sin xy}{\log x}\,dx$ and, with $2\le r<r_1$,

$$F(r,r_1,y)=\int_r^{r_1}\frac{\sin xy}{\log x}\,dx=-\frac{\cos xy}{y\log x}\Big|_r^{r_1}-\frac{1}{y}\int_r^{r_1}\frac{\cos xy}{x(\log x)^2}\,dx. \tag{1.8.80}$$

Therefore

$$|F(r,r_1,y)| \le \frac{1}{y}\left|\frac{2}{\log r}+\int_r^{r_1}\frac{ax}{x(\log x)^2}\right| \le \frac{3}{y\log r}. \tag{1.8.81}$$

Now Theorem 4 implies that $F(y)$ converges uniformly on $[\rho,\infty)$ if $\rho>0$.

(c) Denote $F(y)=\displaystyle\int_0^\infty \frac{\cos xy}{x+y^2}$, and, with $0<r<r_1$,

$$F(r,r_1,y)=\int_r^{r_1}\frac{\cos xy}{x+y^2}\,dx=\frac{1}{y}\left(\frac{\sin xy}{x+y^2}\Big|_r^{r_1}+\int_r^{r_1}\frac{\sin xy}{(x+y^2)^2}\,dx\right), \tag{1.8.82}$$

so

$$|F(r,r_1,y)| \le \frac{3}{y(r+y^2)}. \tag{1.8.83}$$

Now Theorem 4 implies that $F(y)$ converges uniformly on $[\rho,\infty)$ if $\rho>0$.

(d) Denote $F(y)=\displaystyle\int_0^\infty \frac{\sin xy}{1+xy}$, and, with $0<r<r_1$,

$$F(r,r_1,y)=\int_r^{r_1}\frac{\sin xy}{1+xy}\,dx=-\frac{\cos xy}{y(1+xy)}\Big|_r^{r_1}-\int_r^{r_1}\frac{\cos xy}{y^2(1+xy)^2}\,dx, \tag{1.8.84}$$

so

$$|F(r,r_1,y)| \le \frac{3}{y(1+ry)}. \tag{1.8.85}$$

Now Theorem 4 implies that $F(y)$ converges uniformly on $[\rho,\infty)$ if $\rho>0$.

13. Integration by parts yields

$$\begin{aligned}\int_r^{r_1} g(x,y)h(x,y)\,dx &= \int_r^{r_1} g(x,y)H_x(x,y)\,dx\\ &= g(r_1,y)H(r_1,y)-g(r,y)H(r,y)\\ &\quad -\int_r^{r_1} g_x(x,y)H(x,y)\,dx,\end{aligned} \tag{1.8.86}$$

so

$$\begin{aligned}&\left|\int_r^{r_1} g(x,y)h(x,y)\,dx\right| \\ &\le 2\sup_{x\ge r}\left\{\left\{\sup_{y\in S}|g(x,y)H(x,y)|\right\}\right\}+\left|\int_r^{r_1} g_x(x,y)H(x,y)\,dx\right|.\end{aligned} \tag{1.8.87}$$

Now suppose $\epsilon\ge 0$. From our first assmption, there is an $s_0\in[a,b)$ such that

$$\sup_{x\ge r}\left\{\left\{\sup_{y\in S}|g(x,y)H(x,y)|\right\}\right\}<\epsilon, \quad s_0\le r<b. \tag{1.8.88}$$

Since $\int_a^b g_x(x,y)H(x,y)\,dx$ converges uniformly on S, Theorem 4 implies that there is an $r_0\in[a,b)$ such that

$$\left|\int_r^{r_1} g_x(x,y)H(x,y)\,dx\right|\le\epsilon, \quad y\in S, \quad r_0\le r<r_1<b. \tag{1.8.89}$$

Therefore,

$$\left|\int_r^{r_1} g(x,y)h(x,y)\,dx\right|\le 2\epsilon, \quad y\in S, \quad \max(r_0,s_0)\le r<r_1<b. \tag{1.8.90}$$

Now Theorem 4 implies that $\int_a^b g_x(x,y)h(x,y)\,dx$ converges uniformly on S.

14. Theorem 10 *If $f=f(x,y)$ is continuous on $(a,b]\times[c,d]$ and*

$$F(y)=\int_a^b f(x,y)\,dx \tag{A}$$

converges uniformly on $[c,d]$, *then* F *is continuous on* $[c,d]$. *Moreover*,

$$\int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx. \tag{B}$$

We will first show that F in (A) is continuous on $[c,d]$. Since F converges uniformly on $[c,d]$, Definition 1 implies that if $\epsilon > 0$, there is an $r \in (a,b]$ such that

$$\left|\int_a^r f(x,y)\,dx\right| \le \epsilon, \quad c \le y \le d. \tag{1.8.91}$$

Therefore, if y and y_0 are in $[c,d]$, then

$$\begin{aligned} |F(y)-F(y_0)| = & \left|\int_a^b f(x,y)\,dx - \int_a^b f(x,y_0)\,dx\right| \\ \le & \left|\int_r^b [f(x,y)-f(x,y_0)]\,dx\right| + \left|\int_a^r f(x,y)\,dx\right| \\ & + \left|\int_a^r f(x,y_0)\,dx\right| \end{aligned} \tag{1.8.92}$$

so

$$|F(y)-F(y_0)| \le \int_r^b |f(x,y)-f(x,y_0)|\,dx + 2\epsilon. \tag{C}$$

Since f is uniformly continuous on the compact set $[r,b]\times[c,d]$ (Corollary 5.2.14, p. 314), there is a $\delta > 0$ such that

$$|f(x,y)-f(x,y_0)| < \epsilon \tag{1.8.93}$$

if (x,y) and (x,y_0) are in $[r,b]\times[c,d]$ and $|y-y_0| < \delta$. This and (C) imply that

$$|F(y)-F(y_0)| < (r-a)\epsilon + 2\epsilon < (b-a+2)\epsilon \tag{1.8.94}$$

if y and y_0 are in $[c,d]$ and $|y-y_0| < \delta$. Therefore F is continuous on $[c,d]$, so the integral on left side of (B) exists. Denote

$$I = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy. \tag{D}$$

We will show that the improper integral on the right side of (B) converges to I. To this end, denote

$$I(r) = \int_r^b \left(\int_c^d f(x,y)\,dy\right)\,dx. \tag{1.8.95}$$

Since we can reverse the order of integration of the continuous function f over the rectangle $[r,b]\times[c,d]$ (Corollary 7.2.2, p. 466),

$$I(r) = \int_c^d \left(\int_r^b f(x,y)\,dx\right)\,dy. \tag{1.8.96}$$

From this and (D),

$$I - I(r) = \int_c^d \left(\int_a^r f(x,y)\,dx\right)\,dy. \tag{1.8.97}$$

Now suppose $\epsilon > 0$. Since $\int_a^b f(x,y)\,dx$ converges uniformly on $[c,d]$, there is an $r_0 \in (a,b]$ such that

$$\left|\int_a^r f(x,y)\,dx\right| < \epsilon, \quad a < r < r_0, \tag{1.8.98}$$

so $|I-I(r)| < (d-c)\epsilon$, $a < r < r_0$. Hence,

$$\lim_{r\to a+}\int_r^b \left(\int_c^d f(x,y)\,dy\right)\,dx = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy, \tag{1.8.99}$$

which completes the proof of (B).

15. Theorem 11 *Let f and f_y be continuous on $(a,b]\times[c,d]$, and suppose that*

$$F(y)=\int_a^b f(x,y)\,dx \tag{1.8.100}$$

converges for some $y_0\in[c,d]$ and

$$G(y)=\int_a^b f_y(x,y)\,dx \tag{1.8.101}$$

converges uniformly on $[c,d]$. Then F converges uniformly on $[c,d]$ and is given explicitly by

$$F(y)=F(y_0)+\int_{y_0}^y G(t)\,dt,\quad c\le y\le d. \tag{1.8.102}$$

Moreover, F is continuously differentiable on $[c,d]$ and

$$F'(y)=G(y),\quad c\le y\le d, \tag{A}$$

where $F'(c)$ and $f_y(x,c)$ are derivatives from the right, and $F'(d)$ and $f_y(x,d)$ are derivatives from the left.

Let

$$F_r(y)=\int_r^b f(x,y)\,dx,\quad a\le r<b,\quad c\le y\le d. \tag{1.8.103}$$

Since f and f_y are continuous on $[r,b]\times[c,d]$, Theorem 1 implies that

$$F_r'(y)=\int_r^b f_y(x,y)\,dx,\quad c\le y\le d. \tag{1.8.104}$$

Therefore

$$\begin{aligned} F_r(y) &= F_r(y_0)+\int_{y_0}^y\left(\int_r^b f_y(x,t)\,dx\right)dt\\ &= F(y_0)+\int_{y_0}^y G(t)\,dt\\ &\quad +(F_r(y_0)-F(y_0))-\int_{y_0}^y\left(\int_a^r f_y(x,t)\,dx\right)dt,\quad c\le y\le d. \end{aligned} \tag{1.8.105}$$

Therefore,

$$\left|F_r(y)-F(y_0)-\int_{y_0}^y G(t)\,dt\right|\le|F_r(y_0)-F(y_0)|+\left|\int_{y_0}^y\int_a^r f_y(x,t)\,dx\right|dt. \tag{B}$$

Now suppose $\epsilon>0$. Since we have assumed that $\lim_{r\to a+}F_r(y_0)=F(y_0)$ exists, there is an r_0 in (a,b) such that

$$|F_r(y_0)-F(y_0)|<\epsilon,\quad a<r<r_0. \tag{1.8.106}$$

Since we have assumed that $G(y)$ converges for $y\in[c,d]$, there is an $r_1\in(a,b]$ such that

$$\left|\int_a^b f_y(x,t)\,dx\right|<\epsilon,\quad t\in[c,d],\quad a<r\le r_1. \tag{1.8.107}$$

Therefore, (B) yields

$$\left|F_r(y)-F(y_0)-\int_{y_0}^y G(t)\,dt\right|<\epsilon(1+|y-y_0|)\le\epsilon(1+d-c) \tag{1.8.108}$$

if $a<r<\min(r_0,r_1)$ and $t\in[c,d]$. Therefore $F(y)$ converges uniformly on $[c,d]$ and

$$F(y)=F(y_0)+\int_{y_0}^y G(t)\,dt,\quad c\le y\le d. \tag{1.8.109}$$

Since G is continuous on $[c,d]$ by Theorem 10, (A) follows from differentiating this (Theorem 3.3.11, p. 141).

16. Since

$$|f(x)\cos xy| \le |f(x)|, \quad |f(x)\sin xy| \le |f(x)|, \quad \text{and} \quad \int_{-\infty}^{\infty} |f(x)|\,dx < \infty, \tag{1.8.110}$$

Theorems 6 and 7 imply that $\int_{-\infty}^{\infty} f(x)\cos xy\,dx$ and $\int_{-\infty}^{\infty} f(x)\sin xy\,dx$ converge uniformly on $(-\infty,\infty)$, so Theorem 10 implies that $C(y)$ and $S(y)$ are continuous on $(-\infty,\infty)$.

If $y \ne 0$, integrating by parts yields

$$\begin{aligned} C(y) = &\quad f(x)\frac{\sin xy}{y}\Big|_a^{\infty} - \frac{1}{y}\int_a^{\infty} f'(x)\sin xy\,dx \\ = &\quad -f(a)\frac{\sin ay}{y} - \frac{1}{y}\int_a^{\infty} f'(x)\sin xy\,dx \end{aligned} \tag{1.8.111}$$

and

$$\begin{aligned} S(y) = &\quad -f(x)\frac{\cos xy}{y}\Big|_a^{\infty} + \frac{1}{y}\int_a^{\infty} f'(x)\cos xy\,dx \\ = &\quad f(a)\frac{\cos ay}{y} + \frac{1}{y}\int_a^{\infty} f'(x)\cos xy\,dx, \end{aligned} \tag{1.8.112}$$

since $\lim_{x\to\infty} f(x) = 0$. From Exercise [exer:17] with f replaced by f', $\int_1^{\infty} f'(x)\cos xy\,dx$ and $\int_1^{\infty} f'(x)\cos xy\,dx$ are continuous on $(-\infty,\infty)$. Therefore $C(y)$ and $S(y)$ are continuous on $(-\infty,0)\cup(0,\infty)$.

To see that C and S are not necessarily continuous at $y = 0$, let $a = 1$ and $f(x) = 1/x$, so

$$\lim_{x\to\infty} f(x) = 0 \quad \text{and} \quad \int_1^{\infty} |f'(x)| = \int_1^{\infty} \frac{dx}{x^2} = 1. \tag{1.8.113}$$

Then

$$C(y) = \lim_{r\to\infty}\int_1^r \frac{\cos xy}{x}\,dx \quad \text{and} \quad S(y) = \lim_{r\to\infty}\int_1^r \frac{\sin xy}{x}\,dx, \quad y \ne 0. \tag{1.8.114}$$

If $y > 0$ make the change of variable $u = xy$ to see that

$$C(y) = \lim_{r\to\infty}\int_y^{ry} \frac{\cos u}{u}\,du = \int_y^{\infty} \frac{\cos u}{u}\,du \tag{1.8.115}$$

and

$$S(y) = \lim_{r\to\infty}\int_y^{ry} \frac{\sin u}{u}\,du. S(y) = \int_y^{\infty} \frac{\sin u}{u}\,du. \tag{1.8.116}$$

Therefore $\lim_{y\to 0+} C(y) = \infty$, so C is not continuous at $y = 0$. Since $S(0) = 0$ and $\lim_{y\to 0+} S(y) = \int_0^{\infty} \frac{\sin u}{u}\,du \ne 0$, S is not continuous at $y = 0$.

The integral diverges if $y = 0$. If $y \ne 0$ substitute $u = |y|x$ to obtain

$$F(y) = \int_0^{\infty} \frac{dx}{1+x^2y^2} = \frac{1}{|y|}\int_0^{\infty} \frac{du}{1+u^2} = \frac{1}{|y|}\tan^{-1} u\Big|_0^{\infty} = \frac{\pi}{2|y|}, \tag{A}$$

so $F(y)$ converges for all $y \ne 0$. To test for uniform convergence, suppose $|y| > 0$ and $0 < r < r_1$. Then

$$\int_r^{r_1} \frac{dx}{1+x^2y^2} = \frac{1}{|y|}\int_{r|y|}^{r_1|y|} \frac{du}{1+u^2} < \frac{1}{\rho}\int_{r\rho}^{\infty} \frac{du}{1+u^2} \tag{1.8.117}$$

if $|y| \ge \rho$. If $\epsilon > 0$ there is an $\alpha > 0$ such that $\frac{1}{\rho}\int_{\alpha}^{\infty} \frac{du}{1+u^2} < \epsilon$. Therefore $\int_r^{r_1} \frac{dx}{1+x^2y^2} < \epsilon$ if $\alpha/\rho < r < r_1$. Now Theorem 4 implies that $F(y)$ converges uniformly on $(-\infty,-\rho]\cup[\rho,\infty)$ if $\rho > 0$.

To evaluate

$$I = \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x}\,dx, \tag{1.8.118}$$

we note that

$$\frac{\tan^{-1} ax - \tan^{-1} bx}{x} = \int_b^a \frac{dy}{1+x^2y^2}. \tag{1.8.119}$$

Therefore

$$I = \int_0^\infty dx \int_b^a \frac{dy}{1+x^2y^2} = \int_b^a dy \int_0^\infty \frac{dx}{1+x^2y^2} = \frac{\pi}{2}\int_b^a \frac{dy}{y} = \frac{\pi}{2}\log\frac{a}{b}, \tag{1.8.120}$$

where the second equality is valid because of the uniform convergence of $F(y)$ on the closed interval with endpoints a and b, and the third equality follows from (A).

$F(y)$ is a proper integral if $y \ge 0$ and it diverges if $y \le -1$. If $-1 < y < 0$, then

$$F(y) = \int_0^1 x^y\,dx = \frac{x^{y+1}}{y+1}\Big|_0^1 = \frac{1}{y+1} \tag{A}$$

is convergent. Since

$$\int_0^r x^y\,dx = \frac{x^{y+1}}{y+1}\Big|_0^r = \frac{r^{y+1}}{y+1}. \tag{1.8.121}$$

and

$$\frac{\partial}{\partial y}\left(\frac{r^{y+1}}{y+1}\right) = \frac{r^{y+1}}{y+1}\left(\log r - \frac{1}{y+1}\right) < 0 \quad \text{if} \quad 0 < r \le 1 \quad \text{and} \quad y > -1, \tag{1.8.122}$$

it follows that

$$\left|\int_0^r x^y\,dx\right| \le \frac{r^{\rho+1}}{\rho+1} \quad \text{if} \quad 0 < r \le 1 \quad \text{and} \quad -1 < \rho \le y. \tag{1.8.123}$$

Therefore, Theorem 5 implies that $F(y)$ converges uniformly on $[\rho,\infty)$ if $\rho > -1$.

Now Theorem 11 implies that

$$\begin{aligned} I = \int_0^1 \frac{x^a - x^b}{\log x}\,dx &= \int_0^1 dx \int_b^a x^y\,dy = \int_b^a dy \int_0^1 x^y\,dx \\ &= \int_b^a \frac{dy}{y+1} = \log\frac{a+1}{b+1}. \end{aligned} \tag{1.8.124}$$

$F(y) = \displaystyle\int_0^\infty e^{-yx}\cos x\,dx = \frac{y}{y^2+1}$. Since

$$\left|\int_r^\infty e^{-yx}\cos x\,dx\right| \le \int_r^\infty e^{-xy}\,dx = \frac{e^{-yr}}{y}, \tag{1.8.125}$$

Theorem 4 (or Theorem 6) implies that $F(y)$ converges uniformly on $[\rho,\infty)$ if $\rho > 0$. Therefore, Theorem implies that if a, $b > 0$ then

$$\begin{aligned} I = &\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\cos x\,dx = \int_0^\infty \cos x\,dx \int_a^b e^{-yx}\,dy \\ = &\int_a^b dy \int_0^\infty e^{-yx}\cos x\,dx = \int_a^b \frac{y}{y^2+1}\,dy = \frac{1}{2}\log\frac{b^2+1}{a^2+1}. \end{aligned} \tag{1.8.126}$$

$F(y) = \displaystyle\int_0^\infty e^{-yx}\sin x\,dx = \frac{1}{y^2+1}$. Since

$$\left|\int_r^\infty e^{-yx}\sin x\,dx\right| \le \int_r^\infty e^{-yx}\,dx = \frac{e^{-yr}}{y}, \tag{1.8.127}$$

Theorem 4 (or Theorem 6) implies that $F(y)$ converges uniformly on every $[\rho,\infty)$ if $\rho > 0$. Therefore, if a, $b > 0$ then Theorem 11 implies that

$$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\sin x\,dx = \int_0^\infty \sin x\,dx \int_a^b e^{-yx}\,dy \tag{1.8.128}$$

$$= \int_a^b dy \int_0^\infty e^{-yx} \sin x \, dx = \int_a^b \frac{1}{y^2+1}\, dy = \tan^{-1} b - \tan^{-1} a.$$

(e) $F(y) = \displaystyle\int_0^\infty e^{-x} \sin xy \, dx = \frac{y}{y^2+1}$. Since

$$\left| \int_r^\infty e^{-x} \sin xy \, dx \right| \le \int_r^\infty e^{-x} \, dx = e^{-r}, \tag{1.8.129}$$

Theorem 4 (or Theorem 6) implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. Therefore Theorem 11 implies that

$$\begin{aligned} I = & \int_0^\infty e^{-x} \frac{1-\cos ax}{x}\, dx = \int_0^\infty e^{-x}\, dx \int_0^a \sin xy \, dy \\ = & \int_0^a dy \int_0^\infty e^{-x} \sin xy \, dx = \int_0^a \frac{y}{y^2+1}\, dy = \frac{1}{2}\log(1+a^2). \end{aligned} \tag{1.8.130}$$

$F(y) = \displaystyle\int_0^\infty e^{-x} \cos xy \, dx = \frac{1}{y^2+1}$. Since

$$\left| \int_r^\infty e^{-x} \cos xy \, dx \right| \le \int_r^\infty e^{-x} \, dx = e^{-r}, \tag{1.8.131}$$

Theorem 4 (or Theorem 6) implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. Therefore Theorem 11 implies that

$$\begin{aligned} I = & \int_0^\infty e^{-x} \sin ax \, dx = \int_0^\infty e^{-x}\, dx \int_0^a \cos xy \, dy \\ = & \int_0^a dy \int_0^\infty e^{-x} \cos xy \, dx = \int_0^a \frac{1}{y^2+1}\, dy = \tan^{-1} a. \end{aligned} \tag{1.8.132}$$

19. (a) We start with

$$F(y) = \int_0^1 x^y \, dx = \frac{1}{y+1} \quad y > -1. \tag{A}$$

Formally differentiating this yields

$$F^{(n)}(y) = \int_0^1 (\log x)^n x^y \, dx = \frac{(-1)^n n!}{(y+1)^{n+1}}, \quad y > -1. \tag{B}$$

To justify this we will show by induction that the improper integrals

$$I_n(y) = \int_0^1 (\log x)^n x^y \, dx, \quad n = 0, 1, 2, ... \tag{1.8.133}$$

converge uniformly on $[\rho, \infty)$ if $\rho > -1$. We begin with $n = 0$:.

$$\int_0^r x^y \, dx = \frac{x^{y+1}}{y+1}\Big|_{r_1}^r = \frac{r^{y+1}}{y+1} \le \frac{r^{y+1}}{\rho+1}, \quad -1 < \rho \le y. \tag{1.8.134}$$

so $I_0(y) = F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > -1$. Now suppose that $I_n(y)$ converges uniformly on $[\rho, \infty)$. Integrating by parts yields

$$\begin{aligned} \int_{r_1}^r (\log x)^{n+1} x^y \, dx = & \frac{r^{y+1}(\log r)^{n+1} - r_1^{y+1}(\log r_1)^{n+1}}{y+1} \\ & -\frac{n+1}{y+1} \int_{r_1}^r (\log x)^n x^y \, dx, \quad -1 < y < \infty. \end{aligned} \tag{1.8.135}$$

Letting $r_1 \to 0$ yields

$$\int_0^r (\log x)^{n+1} x^y \, dx = \frac{r^{y+1}(\log r)^{n+1}}{y+1} - \frac{n+1}{y+1} \int_0^r (\log x)^n x^y \, dx. \tag{C}$$

Since the integral on the right converges, it follows that the integral on the left converges; in fact

$$\int_0^1 (\log x)^{n+1} x^y \, dx = -\frac{n+1}{y+1} \int_0^1 (\log x)^n x^y \, dx. \tag{1.8.136}$$

We must still show that the integral on the left converges uniformly on $[\rho,\infty)$ if $\rho>-1$. To this end, note from (C) that

$$\left|\int_0^r (\log x)^{n+1}x^y\,dx\right|\le\left|\frac{r^{\rho+1}(\log r)^{n+1}}{\rho+1}\right|+\frac{n+1}{\rho+1}\left|\int_0^r (\log x)^n x^y\,dx\right| \tag{D}$$

if $y\ge\rho$, Now suppose $\epsilon>0$. Since $\lim\limits_{r\to0+} r^{\rho+1}(\log r)^{n+1}=0$, there is an $r_1\in(0,1)$ such that

$$\left|\frac{r^{\rho+1}(\log r)^{n+1}}{\rho+1}\right|\le\frac{\epsilon}{2}\quad\text{if}\quad 0<r<r_1. \tag{1.8.137}$$

Since $\int_0^r (\log x)^n x^y\,dx$ is uniformly convergent (by our induction assumption), there is $r_2\in(0,1)$ such that

$$\frac{n+1}{\rho+1}\left|\int_0^r (\log x)^n x^y\,dx\right|\le\frac{\epsilon}{2},\quad y\ge\rho, \tag{1.8.138}$$

Now (D) implies that

$$\left|\int_0^r (\log x)^{n+1}x^y\,dx\right|\epsilon,\quad y\in[\rho,\infty),\quad 0<r<\min(r_1,r_2). \tag{1.8.139}$$

This, Theorem 11, and an easy induction argument imply (B).

(b) Substituting $x=u\sqrt{y}$ yields

$$F(y)=\int_0^\infty\frac{dx}{x^2+y}=\frac{1}{\sqrt{y}}\int_0^\infty\frac{du}{u^2+1}=\frac{\pi}{2\sqrt{y}},\quad y>0. \tag{A}$$

Formally differentiating this yields yields

$$\begin{aligned}\int_0^\infty\frac{dx}{(x^2+y)^{n+1}}&=\frac{\pi}{2n+1}1\cdot3\cdots(2n-1)y^{-n-1/2}=\frac{\pi}{2^{2n+1}}\frac{(2n)!}{n!}y^{-n-1/2}\\&=\frac{\pi}{2^{2n+1}}\binom{2n}{n}y^{-n-1/2},\quad y>0.\end{aligned} \tag{1.8.140}$$

Theorem 11 implies that the formal differentiation is legitimate, since, if $y\ge0$ and $r>0$, then

$$\int_r^\infty\frac{dx}{(x^2+y)^{n+1}}\le\int_r^\infty x^{-2n-2}dx=\frac{r^{-2n-1}}{(2n-1)}; \tag{1.8.141}$$

hence, the improper integrals $\displaystyle\int_0^\infty\frac{dx}{(x^2+y)^{n+1}}$, $n=0,1,2,\dots$ converge uniformly on $[0,\infty)$.

Denote $I_n(y)=\displaystyle\int_0^\infty x^{2n+1}e^{-yx^2}\,dx$. Then

$$I_0(y)=\int_0^\infty xe^{-yx^2}=\frac{1}{2}\int_0^\infty 2xe^{-yx^2}\,dx=-\frac{1}{2y}e^{-yx^2}\Big|_0^\infty=\frac{1}{2y}. \tag{1.8.142}$$

Since

$$\int_r^\infty x^{2n+1}e^{-yx^2}\,dx\le\int_r^\infty x^{2n+1}e^{-\rho x^2}\,dx\quad\text{if}\quad 0<\rho\le r, \tag{1.8.143}$$

if $n\ge0$, we can differentiate I_n formally with respect to $y\in(0,\infty)$ to obtain

$$I_n(y)=(-1)^nI_0^{(n)}=\frac{n!}{2y^{n+1}}. \tag{1.8.144}$$

(d) Denote

$$\begin{aligned}I(y)&=\int_0^\infty y^x\,dx=\int_0^\infty e^{x\log y}\,dx=\frac{1}{\log y}\int_0^\infty(\log y)y^x\,dx\\&=\frac{y^x}{\log y}\Big|_0^\infty=-\frac{1}{\log y}\quad 0<y<1.\end{aligned} \tag{1.8.145}$$

Formally differentiating this yields $I'(y)=\int_0^\infty xy^{x-1}\,dx$. There are two improper integrals here: $J_1(y)=\int_0^1 xy^{x-1}\,dx$ and $J_2(y)=\int_1^\infty xy^{x-1}\,dx$. If $r<1$ then

$$\int_0^r xy^{x-1}\,dx=\frac{1}{y}\int_0^r xy^x\,dx\le\frac{1}{y}\int_0^r x\,dx=\frac{r^2}{2y}\le\frac{r^2}{2\rho_1},\quad 0<\rho_1\le y\le 1. \tag{1.8.146}$$

Therefore $J_1(y)$ converges uniformly on $[\rho_1,1]$. If $x>r>1$ and $\rho_2<1$ then

$$\int_r^\infty xy^{x-1}\,dx<\int_r^\infty x\rho_2^{x-1}\,dx=\frac{1}{\rho_2}\int_r^\infty x\rho_2^x\,dx, \tag{1.8.147}$$

Since

$$\lim_{x\to\infty}\frac{1}{\rho_2}\int_r^\infty x\rho_2^x\,dx=0 \tag{1.8.148}$$

Theorem 7 implies that $J_2(y)$ converges uniformly on $[0,\rho_2]$. Therefore, if $0<\rho_1<\rho_2<1$ then $\int_0^\infty xy^{x-1}$ converges uniformly on $[\rho_1,\rho_2]$. Now Theorem 11 implies that

$$I'(y)=\int_0^\infty xy^{x-1}\,dx,\quad 0<y<1. \tag{A}$$

However, since $I(y)=-\dfrac{1}{\log y}$, we know that $I'(y)=\dfrac{1}{y(\log y)^2}$. This and (A) imply that $\int_0^\infty xy^x\,dx=\dfrac{1}{(\log x)^2}$.

20. Here $F(y)=\int_0^\infty e^{-x^2}\cos 2xy\,dx$, so Theorem 11 implies that

$$F'(y)=-2\int_0^\infty xe^{-x^2}\sin 2xy\,dx, \tag{A}$$

since the integral on the right converges uniformly on $(-\infty,\infty)$, by Theorem 6.

Integration by parts yields

$$\begin{aligned} F(y)&=\frac{1}{2y}\int_0^\infty e^{-x^2}(2y\cos 2xy)\,dx\\ &=\frac{1}{2y}\left(e^{-x^2}\sin 2xy\,dx\Big|_0^\infty+2\int_0^\infty xe^{-x^2}\sin 2xy\,dx\right)\\ &=\frac{1}{y}\int_0^\infty xe^{-x^2}\sin 2xy\,dx=-\frac{1}{2y}F'(y). \end{aligned} \tag{1.8.149}$$

From this and (A), $F'(y)+2yF(y)=0$, so $\dfrac{F'(y)}{F(y)}=-2y$, $\log F(y)=-y^2+\log F(0)$, and $F(y)=F(0)e^{-y^2}$. Since $F(0)=\int_0^\infty e^{-x^2}\,dx=\dfrac{\sqrt{\pi}}{2}$ (Example 12), $F(y)=\dfrac{\sqrt{\pi}}{2}e^{-y^2}$.

Here $F(y)=\int_0^\infty e^{-x^2}\sin 2xy\,dx$, so Theorem 11 implies that

$$F'(y)=2\int_0^\infty xe^{-x^2}\cos 2xy\,dx, \tag{A}$$

since the integral on the right converges uniformly on $(-\infty,\infty)$. Integrating this by parts yields

$$\begin{aligned} F'(y)&=-e^{-x^2}\cos 2xy\Big|_0^\infty-2y\int_0^\infty e^{-x^2}\sin 2xy\,dx\\ &=1-2yF(y), \end{aligned} \tag{1.8.150}$$

so $F'(y)+2yF(y)=1$, $e^{y^2}F'(y)+2e^{y^2}yF(y)=e^{y^2}$, and $\left(e^{y^2}F(y)\right)'=e^{y^2}$. Therefore, since $F(0)=0$, $F(y)=e^{-y^2}\int_0^y e^{u^2}\,du$.

22. Theorems 6 and 11 imply that S and C are n times continuously differentiable on $(-\infty,\infty)$ if $\int_{-\infty}^\infty |x^nf(x)|\,dx<\infty$.

23. We will show first that

$$C_k(y)=\int_0^\infty x^kf(x)\cos xy\,dx\ \text{ and }\ S_k(y)=\int_0^\infty x^kf(x)\sin xy\,dx,\ 0<k<n \tag{1.8.151}$$

converge uniformly on $U_\rho = (-\infty,-\rho]\cup[\rho,\infty)$ if $\rho>0$. Note that if $\lim_{x\to\infty} x^n f(x) = 0$, then $\lim_{x\to\infty} x^k f(x) = 0$, $k = 0,\ 1,\ 2,\ldots n$. If $0 \le k \le n$, then

$$\int_r^{r_1} x^k f(x)\cos xy\,dx = \frac{1}{y}\left[x^k f(x)\sin xy\Big|_r^{r_1} - \int_r^{r_1}(x^k f(x))'\sin xy\,dx\right]. \tag{A}$$

Henceforth k is fixed. Our assumptions imply that if $\rho>0$ and $\epsilon>0$ then there is an $r_0\in[a,\infty)$ such that

$$\int_{r_0}^{\infty}|(x^k f(x))'|\,dx<\rho\epsilon \quad\text{and}\quad |x^k f(x)|<\rho\epsilon,\quad x\ge r_0. \tag{1.8.152}$$

Therefore (A) implies that

$$\left|\int_r^{r_1} x^k f(x)\cos xy\,dx\right|<3\epsilon,\quad r_0\le r<r_1,\ y\in(-\infty,-\rho]\cup[\rho,\infty). \tag{1.8.153}$$

Now Theorem 4 implies that C_0, C_1,..., C_k converge uniformly on $(-\infty,-\rho]\cup[\rho,\infty)$. Since every $y\ne0$ is in such an interval, Theorem 11 now implies that that if $y\ne0$ then

$$C^{(k)}(y) = \int_a^{\infty} x^k f(x)\sin xy\,dx,\quad 0\le k\le n. \tag{1.8.154}$$

A similar argument applies to S, S',...$S^{(n)}$.

Let $I(y;r,r_1) = \displaystyle\int_r^{r_1}\frac{1}{x}\sin\frac{y}{x}\,dx$. Assume for the moment that $y\ge0$. Substituting $u=y/x$ yields

$$I(y;r,r_1) = \int_{y/r_1}^{y/r}\left(\frac{u}{y}\right)\sin u\left(-\frac{y}{u^2}\right)\,du = \int_{y/r_1}^{y/r}\frac{\sin u}{u}\,du. \tag{1.8.155}$$

Therefore, since $\left|\dfrac{\sin u}{u}\right|\le1$ for all u, $|I(y;r,r_1)|\le y/r$,$1\le r\le r_1$. In fact, since $I(-y;r,r_1) = -I(y;r,r_1)$, we can write $|I(y;r,r_1)| \le |y|/r$, $1\le r\le r_1$. Therefore, Theorem 4 implies that $\displaystyle\int_1^{\infty}\frac{1}{x}\sin\frac{y}{x}\,dx$ converges uniformly on every finite interval.

Now denote $F_r(y) = \displaystyle\int_1^r\cos\frac{y}{x}\,dx$. substituting $u=y/x$ yields $F_r(y) = y\displaystyle\int_{y/r}^{y}\frac{\cos u}{u^2}\,du$, so $\lim_{r\to\infty}F_r(y)=\infty$ for all $y\ge0$. Since $F_r(-y)$ $= F_r(y)$, it follows that $\lim_{r\to\infty}F_r(y)=\infty$ for all y, so the answer to the question is “no.”

Let P_n be the induction assumption

$$F^{(n)}(s) = (-1)^n\int_0^{\infty}e^{-sx}x^n f(x)\,dx,\quad s>s_0, \tag{1.8.156}$$

which is true by the definition of F for $n=0$. If P_n is true, then Theorems 11 and 13 imply that

$$\begin{aligned} F^{(n+1)}(s) = &\quad (-1)^n\frac{d}{ds}\int_0^{\infty}e^{-sx}x^n f(x)\,dx = (-1)^n\int_0^{\infty}\frac{d}{ds}\left(e^{-sx}x^n f(x)\right)\,dx\\ = &\quad (-1)^{n+1}\int_0^{\infty}e^{-sx}x^{n+1}f(x)\,dx, \end{aligned} \tag{1.8.157}$$

so P_n implies P_{n+1}, which completes the induction proof.

Let $G(x) = \displaystyle\int_0^x e^{-s_0t}f(t)\,dt$. If $s>s_0$ then

$$F(s) = \int_0^{\infty}e^{-sx}f(x)\,dx = \int_0^{\infty}e^{-(s-s_0)x}G'(x)\,dx = (s-s_0)\int_0^{\infty}e^{-(s-s_0)x}G(x)\,dx \tag{A}$$

(integration by parts). Since $(s-s_0)\displaystyle\int_0^{\infty}e^{-(s-s_0)x}\,dx = 1$, (A) implies that

$$F(s)-F(s_0) = (s-s_0)\int_0^{\infty}e^{-(s-s_0)x}(G(x)-F(s_0))\,dx. \tag{B}$$

https://math.libretexts.org/@go/page/17329

Now suppose $\epsilon > 0$. Since $F(s_0) = \int_0^\infty e^{-s_0 t} f(t)\,dt = \lim_{t\to\infty} G(x)$, there is an r such that $|G(x) - F(s_0)| < \epsilon$ if $x \ge r$; hence, from (B), then

$$\begin{aligned} |F(s) - F(s_0)| \le & \quad (s-s_0)\int_0^r e^{-(s-s_0)x}|G(x) - F(s_0)| + \epsilon(s-s_0)\int_r^\infty e^{-(s-s_0)x}\,dx \\ < & \quad (s-s_0)\int_0^r e^{-(s-s_0)x}|G(x) - F(s_0)| + \epsilon. \end{aligned} \tag{1.8.158}$$

Since r is fixed, we can let $s \to s_0^+$ to conclude that $\limsup_{s\to s_0+} |F(s) - F(s_0)| \le \epsilon$, which implies that $\lim_{s\to S_0+} F(s) = F(s_0)$.

If $s \ge s_1 > s_0$ then

$$|e^{-sx} f(x)| = |e^{-(s-s_0)x} e^{s_0 x} f(x)| \le M e^{-(s-s_0)x} \le M e^{-(s_1-s_0)x}. \tag{1.8.159}$$

Since

$$\int_0^\infty M e^{-(s_1-s_0)x}\,dx = \frac{M}{s_1 - s_0} < \infty, \tag{1.8.160}$$

Theorem 6 implies the stated conclusion.

In Theorem 13 we assumed only that $\int_0^x e^{-s_0 u} f(u)\,du$ is bounded; here we are assuming that $\int_0^\infty e^{-s_0 u} f(u)\,du$ is convergent.

Let

$$G(x) = \int_x^\infty e^{-s_0 t} f(t)\,dt \quad \text{and} \quad H(x) = \sup\left\{|G(t)|\,\middle|\, t \ge x\right\}. \tag{1.8.161}$$

Then

$$|G(x)| \le H(x) \quad \text{and} \quad \lim_{x\to\infty} G(x) = \lim_{x\to\infty} H(x) = 0, \tag{A}$$

since $\int_0^\infty e^{-s_0 x} f(x)\,dx$ converges. Since f is continuous on $[0,\infty)$, $G'(x) = -e^{-s_0 x} f(x)$. Integration by parts yields

$$\begin{aligned} \int_r^\infty e^{-sx} f(x)\,dx = & \quad \int_r^\infty e^{-(s-s_0)x}(e^{-s_0 x} f(x))\,dx = -\int_0^\infty e^{-(s-s_0)x} G'(x)\,dx \\ = & \quad -e^{-(s-s_0)x} G(x)\Big|_r^\infty + (s-s_0)\int_r^\infty e^{-(s-s_0)x} G(x)\,dx \\ = & \quad e^{-(s-s_0)r} G(r) + (s-s_0)\int_r^\infty e^{-(s-s_0)x} G(x)\,dx, \quad s \ge s_0. \end{aligned} \tag{1.8.162}$$

Therefore

$$\begin{aligned} \left|\int_r^\infty e^{-sx} f(x)\,dx\right| \le & \quad |G(r)| e^{-(s-s_0)r} + H(r)(s-s_0)\int_r^\infty e^{-(s-s_0)x}\,dx \\ = & \quad (G(r) + H(r)) e^{-(s-s_0)r} \le 2H(r) e^{-(s-s_0)}, \quad s \ge s_0, \end{aligned} \tag{1.8.163}$$

so (A) implies that $F(s)$ converges uniformly on $[s_0,\infty)$.

From Theorem 13, $F(s) = \int_0^\infty e^{-sx} f(x)\,dx$ converges for all $s > s_0$. Denote $G(x) = \int_0^x e^{-s_0 t} f(t)\,dt$, $x \ge 0$. Then

$$\begin{aligned} F(s) = & \quad \int_0^\infty e^{-(s-s_0)x} e^{-s_0 x} f(x)\,dx = \int_0^\infty e^{-(s-s_0)x} G'(x)\,dx \\ = & \quad (s-s_0)\int_0^\infty e^{-(s-s_0)x} G(x)\,dx \end{aligned} \tag{1.8.164}$$

(integration by parts). Since $(s-s_0)\int_0^\infty e^{-(s-s_0)x}\,dx = 1$,

$$F(s) - F(s_0) = \int_0^\infty e^{-(s-s_0)x}(G(x) - F(s_0))\,dx \tag{1.8.165}$$

If $\epsilon > 0$ there is an R such that $|G(x) - F(s_0)| < \epsilon$ if $x \ge R$. Therefore, if $s > s_0$ then

$$|F(s) - F(s_0)| < \quad (s-s_0)\int_0^R e^{-(s-s_0)x}|G(x) - F(s_0)|\,dx + \epsilon \tag{1.8.166}$$

$$< (s-s_0)\int_0^{\infty} |G(x)-F(s_0)|\,dx+\epsilon.$$

Hence $\limsup_{s\to s_0+}|F(s)-F(s_0)|\le\epsilon$. Since ϵ is arbitrary, this implies that $\lim_{s\to s_0+}|F(s)-F(s_0)|=0$.

The assumptions of Exercise 28 imply that $\int_r^{\infty} e^{-s_0x}f(x)\,dx$ converges for every $r>0$. Since

$$\int_r^{\infty} e^{-s_0x}f(x)\,dx=\int_0^{\infty} e^{-s(r+x)}f(x+r)\,dx=e^{-sr}\int_0^{\infty} e^{-sx}f(x+r)\,dx, \tag{1.8.167}$$

we can apply the result of Exercise 30 with $f(x)$ replaced by $f(x+r)$, to conclude that

$$\begin{aligned}\lim_{s\to s_0+}\int_r^{\infty} e^{-sx}f(x)\,dx &= e^{-s_0r}\int_0^{\infty} e^{-s_0x}f(x+r)\,dx\\ &= \int_0^{\infty} e^{-s_0(x+r)}f(x+r)\,dx\\ &= \int_r^{\infty} e^{-s_0x}f(x)\,dx.\end{aligned} \tag{1.8.168}$$

If $G(x)=\int_0^x e^{-s_0t}f(t)\,dt$, then $|G(x)|\le M$ on $[0,\infty)$ for some M. If $\epsilon>0$, there is an $r>0$ such that

$$\int_0^r e^{-s_0x}|f(x)|\,dx<\epsilon. \tag{A}$$

If $s>s_0$, then

$$\begin{aligned}\int_r^{\infty} e^{-sx}f(x)\,dx &= \int_r^{\infty} e^{-(s-s_0)x}G'(x)\,dx\\ &= e^{-(s-s_0)x}G(x)\Big|_r^{\infty}+(s-s_0)\int_r^{\infty} G(x)e^{-(s-s_0)x}\,dx\\ &= -e^{-(s-s_0)r}G(r)+(s-s_0)\int_r^{\infty} G(x)e^{-(s-s_0)x}\,dx.\end{aligned} \tag{1.8.169}$$

Therefore, since $|G(x)|\le M$,

$$\begin{aligned}\left|\int_r^{\infty} e^{-sx}f(x)\,dx\right| &\le Me^{-(s-s_0)r}+M(s-s_0)\int_r^{\infty} e^{-(s-s_0)x}\,dx\\ &= M\left(e^{-(s-s_0)r}-e^{-(s-s_0)x}\Big|_r^{\infty}\right)=2Me^{-(s-s_0)r}.\end{aligned} \tag{1.8.170}$$

This and (A) imply that

$$\left|\int_0^{\infty} e^{-sx}f(x)\,dx\right|\le\epsilon+2Me^{-(s-s_0)r}. \tag{1.8.171}$$

Therefore,

$$\limsup_{s\to\infty}\left|\int_0^{\infty} e^{-sx}f(x)\,dx\right|\le\epsilon. \tag{1.8.172}$$

Since ϵ is arbitrary, this implies that

$$\limsup_{s\to\infty}\int_0^{\infty} e^{-sx}f(x)\,dx=0, \tag{1.8.173}$$

From Exercise 18**(d)**, $\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}\sin x\,dx=\tan^{-1}b-\tan^{-1}a$. From Exercise 30, letting $b\to\infty$ yields

$$\int_0^{\infty} e^{-ax}\frac{\sin x}{x}\,dx=\frac{\pi}{2}-\tan^{-1}a,\quad\text{so }\mathbf{(b)}\quad\int_0^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}. \tag{1.8.174}$$

Integrating by parts yields

$$\int_0^r e^{-sx}f'(x)\,dt=e^{-sr}f(r)-f(0)+\int_0^r se^{-sx}f(x)\,dx. \tag{A}$$

Suppose $s \ge s_1 > s_0$. Since $|f(x)| \le Me^{s_0x}$, $e^{-sr}|f(r)| \le Me^{-(s_1-s_0)r}$. Therefore $e^{-sr}f(r) = 0$ converges uniformly to zero on $[s_1, \infty)$. Since

$$\begin{aligned} \left|\int_r^\infty se^{-sx}f(x)\,dx\right| \le & \qquad M|s|\int_r^\infty e^{-(s-s_0)x}\,dx \le \frac{M|s|e^{-(s_1-s_0)r}}{s-s_0} \\ \le & \quad \frac{M(s-s_0+|s_0|)e^{-(s_1-s_0)r}}{s-s_0} \le M\left(1+\frac{|s_0|}{s_1-s_0}\right)e^{-(s_1-s_0)r}, \end{aligned} \tag{1.8.175}$$

it follows that $\int_r^\infty se^{-sx}f(x)\,dx$ converges to zero uniformly on $[s_1, \infty)$. Since this implies that $\int_0^r se^{-sx}f(x)\,dx$ converges uniformly on $[s_1, \infty)$, (A) implies that $G(s)$ converges uniformly on $[s_1, \infty)$.

(b) In this case let $f'(x) = xe^{x^2}\sin e^{x^2}$, so $f(x) = -\frac{1}{2}\cos e^{x^2}$. Since $|\cos e^{x^2}| \le 1$ for all x, the hypotheses stated in (a) hold with $s_0 = 0$. Therefore $G(s)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

33. We will first show that $\int_0^\infty e^{-s_0x}\frac{f(x)}{x}\,dx$ converges. Denote $G(x) = \int_0^x e^{-s_0t}f(t)\,dt$. Since $F(s_0)$ is convergent; say $|G(x)| \le M$, $0 \le x < \infty$. If $0 < r < r_1$ then

$$\int_r^{r_1} e^{-s_0x}\frac{f(x)}{x}\,dx = \int_r^{r_1}\frac{G'(x)}{x}\,dx = \frac{G(r)}{r} - \frac{G(r_1)}{r_1} - \int_r^{r_1}\frac{G(x)}{x^2}\,dx. \tag{1.8.176}$$

Therefore

$$\left|\int_r^{r_1} e^{-s_0x}\frac{f(x)}{x}\,dx\right| \le \frac{3M}{\rho}, \quad \rho < r < r_1 \tag{1.8.177}$$

so Theorem 2 implies that $H(s) = \int_0^\infty e^{-st}\frac{f(x)}{x}\,dx$ converge when $s = s_0$. Therefore Exercise 27 implies that it converges uniformly on $[s_0, \infty)$, Therefore Theorem 10 implies that

$$\begin{aligned} \int_{s_0}^s F(u)\,du = & \qquad \int_{s_0}^s\left(\int_0^\infty e^{-ux}f(x)\,dx\right)du = \int_0^\infty\left(\int_{s_0}^s e^{-ux}\,du\right)f(x)\,dx \\ = & \qquad \int_0^\infty (e^{-s_0x} - e^{-sx})\frac{f(x)}{x}\,dx \end{aligned} \tag{1.8.178}$$

From Exercise 30 (with $f(x)$ replaced by $f(x)/x$), $\lim_{s\to\infty}\int_0^\infty e^{-sx}\frac{f(x)}{x}\,dx = 0$, which implies the stated conclusion.

This page titled 1.8: Improper Functions (Exercises) is shared under a CC BY-NC-SA 3.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

Index

D

Dirichlet’s Tests
1.5: Dirichlet’s Tests

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: Functions Defined by Improper Integrals (Trench)

Webpages: 19

Applicable Restrictions: Noncommercial

All licenses found:

- Undeclared: 52.6% (10 pages)
- CC BY-NC-SA 3.0: 47.4% (9 pages)

By Page

- Functions Defined by Improper Integrals (Trench) - *CC BY-NC-SA 3.0*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Chapters - *Undeclared*
    - 1.1: Introduction to Improper Functions - *CC BY-NC-SA 3.0*
    - 1.2: Preparation - *CC BY-NC-SA 3.0*
    - 1.3: Uniform convergence of improper integrals - *CC BY-NC-SA 3.0*
    - 1.4: Absolutely Uniformly Convergent Improper Integrals - *CC BY-NC-SA 3.0*
    - 1.5: Dirichlet's Tests - *CC BY-NC-SA 3.0*
    - 1.6: Consequences of Uniform Convergence - *CC BY-NC-SA 3.0*
    - 1.7: Applications to Laplace transforms - *CC BY-NC-SA 3.0*
    - 1.8: Improper Functions (Exercises) - *CC BY-NC-SA 3.0*
  - Back Matter - *Undeclared*
    - Index - *Undeclared*
    - Glossary - *Undeclared*
    - Detailed Licensing - *Undeclared*